Chapter 3

Physics of Josephson Junctions: The Voltage State
3. Physics of Josephson junctions: The voltage state

For bias currents $I > I_s^m$

→ Finite junction voltage
→ Phase difference $\varphi$ evolves in time: $\frac{d\varphi}{dt} \propto V$
→ Finite voltage state of the junction corresponds to a dynamic state

→ Only part of the total current is carried by the Josephson current
   → additional resistive channel
   → capacitive channel
   → noise channel

Key questions
→ How does the phase dynamics look like?
→ Current-voltage characteristics for $I > I_s^m$?
→ What is the influence of the resistive damping?
At finite temperature $T > 0$

→ Finite density of “normal” electrons
→ Quasiparticles
→ Zero-voltage state: No quasiparticle current
→ For $V > 0$ → Quasiparticle current = Normal current $I_N$ → Resistive state

High temperatures close to $T_c$

→ For $T \approx T_c$ and $2\Delta(T) \ll k_B T$: (almost) all Cooper pairs are broken up
→ Ohmic current-voltage characteristic (IVC)
→ $I_N = G_N V$, where $G_N \equiv \frac{1}{R_N}$ is the normal conductance

Large voltage $V > V_g = \frac{\Delta_1 + \Delta_2}{e}$

→ External circuit provides energy to break up Cooper pairs
→ Ohmic IVC

For $T \ll T_c$ and $|V| < V_g$

→ Vanishing quasiparticle density → No normal current
3.1.1 The normal current: Junction resistance

Current-voltage characteristic

For \( T \ll T_c \) and \( |V| < V_g \)

\( \rightarrow \) IVC depends on sweep direction and on bias type (current/voltage)

\( \rightarrow \) Hysteretic behavior

\( \rightarrow \) Current bias \( I = I_s + I_N = \text{const} \).

Voltage state

\( \rightarrow \) Bias current \( I \)

\( \rightarrow \) \( I_s(t) = I_c \sin \varphi(t) \) is time dependent

\( \rightarrow \) \( I_N \) is time dependent

\( \rightarrow \) Junction voltage \( V = \frac{I_N}{G_N} \) is time dependent

\( \rightarrow \) IVC shows time-averaged voltage \( \langle V \rangle \)

Equivalent conductance \( G_N \) at \( T = 0 \):

\[
G_N(V) = \begin{cases} 
0 & \text{for } |V| < 2\Delta/e \\
\frac{1}{R_N} & \text{for } |V| \geq 2\Delta/e 
\end{cases}
\]

Circuit model

\( I = I_c \sin \varphi \)

\( V = \frac{\varphi_B}{2\pi} \frac{d\varphi}{dt} \)

\( G_N(V) \)

\( C \)

\( I_F \)
3.1.1 The normal current: Junction resistance

Finite temperature

→ Sub-gap resistance $R_{sg}(T)$ for $|V| < V_g$
→ $R_{sg}(T)$ determined by amount of thermally excited quasiparticles

$$G_{sg}(T) = \frac{1}{R_{sg}(T)} = \frac{n(T)}{n_{tot}} G_N \quad n(T) \rightarrow \text{Density of excited quasiparticles}$$

→ for $T > 0$ we get

$$G_N(V, T) = \begin{cases} \frac{1}{R_{sg}(T)} & \text{for } |V| < 2\Delta(T)/e \\ \frac{1}{R_N} & \text{for } |V| \geq 2\Delta(T)/e \end{cases} \rightarrow \text{Nonlinear conductance } G_N(V, T)$$

→ Characteristic voltage ($I_cR_N$-product)

$$V_c \equiv I_cR_N = \frac{I_c}{G_N}$$

Note:
→ There may be a frequency dependence of the normal channel
→ Normal channel depends on junction type
3.1.2 The displacement current: Junction capacitance

If $\frac{dV}{dt} \neq 0 \Rightarrow$ Finite displacement current

\[ I_D = C \frac{dV}{dt} \]

$\Rightarrow C \leftrightarrow$ junction capacitance

$\Rightarrow$ For planar tunnel junction

\[ C = \frac{\varepsilon_0 A_i}{d} \]

$\Rightarrow$ Additional current channel

With $V = L_c \frac{dI_s}{dt}$, $I_N = V G_N$, $I_D = C \frac{dV}{dt}$, $L_s = \frac{L_c}{\cos \varphi} \geq L_c$ and $G_N(V, T) = \frac{1}{R_N}$

\[ L_c = \frac{\hbar}{2eI_c} \quad \text{Josephson inductance} \]

\[ I_s \leq \frac{V}{\omega L_c} \quad I_N \leq \frac{V}{R_N} \quad I_D \approx \omega C V \]
3.1.3 Characteristic times and frequencies

**Characteristic frequencies**

Equivalent parallel $LRC$ circuit

$\rightarrow L_c, R_N, C$

$\rightarrow$ Three characteristic frequencies

**Plasma frequency**

$$\omega_p = \frac{1}{\tau_p} \equiv \frac{1}{\sqrt{L_c C}} = \sqrt{\frac{2eI_c}{\hbar C}}$$

$\rightarrow \omega_p \propto \sqrt{\frac{I_c}{C_A}}$, where $C_A \equiv \frac{C}{A}$ is the specific junction capacitance

$\rightarrow \omega < \omega_p \rightarrow I_D < I_s$

**Inductive $L_c/R_N$ time constant**

$$\omega_c = \frac{1}{\tau_c} \equiv \frac{R_N}{L_c} = \frac{2e}{\hbar} V_c = \frac{2\pi}{\Phi_0} V_c$$

$V_c = I_c R_N$

$\rightarrow$ Inverse relaxation time in the normal+supercurrent system

$\rightarrow$ $\omega_c$ follows from $V_c$ ($2$nd Josephson equation)

$\rightarrow I_N < I_c$ for $V < V_c$ or $\omega < \omega_c$

**Capacitive $R_N C$ time constant**

$$\omega_{RC} = \frac{1}{\tau_{RC}} \equiv \frac{1}{R_N C} = \frac{\omega_p^2}{\omega_c}$$

$\rightarrow I_D < I_N$ for $\omega < \frac{1}{\tau_{RC}}$
3.1.3 Characteristic times and frequencies

Stewart-McCumber parameter and quality factor

→ Stewart-McCumber parameter
\[ \beta_C = \frac{\omega_C^2}{\omega_p^2} = \frac{\omega_c}{\omega_{RC}} = \omega_c \tau_{RC} = \frac{2e}{\hbar} l_c R_N^2 C \]

→ Quality factor
\[ Q = \frac{RC}{\sqrt{LC}} = \frac{\omega_p}{\omega_{RC}} = \frac{\omega_c}{\omega_p} = \sqrt{\beta_C} \]

(Q compares the decay of oscillation amplitudes to the oscillation period)

Limiting cases

→ \( \beta_C \ll 1 \)
  → Small capacitance and/or small resistance
  → Small \( R_N C \) time constants \((\tau_{RC} \omega_p \ll 1)\)
  → Highly damped (overdamped) junctions

→ \( \beta_C \gg 1 \)
  → Large capacitance and/or large resistance
  → Large \( R_N C \) time constants \((\tau_{RC} \omega_p \gg 1)\)
  → Weakly damped (underdamped) junctions
3.1.4 The fluctuation current

Fluctuation/noise

→ Langefoin method: include random source → fluctuating noise current
→ type of fluctuations: thermal noise, shot noise, 1/f noise

Thermal Noise

Johnson-Nyquist formula for thermal noise ($k_B T \gg eV, \hbar \omega$):

$$S_I(f) = \frac{4k_B T}{R_N} \quad \text{(current noise power spectral density)}$$

$$S_V(f) = 4k_B T R_N \quad \text{(voltage noise power spectral density)}$$

Relative noise intensity (thermal energy/Josephson coupling energy):

$$\gamma \equiv \frac{k_B T}{E_J} = \frac{2e}{\hbar} \frac{k_B T}{I_c} \quad \Rightarrow \gamma \equiv \frac{I_T}{I_c} \quad \text{with} \quad I_T = \frac{2e}{\hbar} k_B T$$

$I_T \equiv$ thermal noise current

$T = 4.2 \text{ K} \rightarrow I_T \approx 0.15 \mu A$
3.1.4 The fluctuation current

**Shot Noise**

Schottky formula for shot noise \((eV \gg k_B T \Rightarrow V > 0.5 \text{ mV} @ 4.2 \text{ K})\):
\[
S_I(f) = 2eI_N
\]

\(\rightarrow\) Random fluctuations due to the discreteness of charge carriers  
\(\rightarrow\) Poisson process \(\rightarrow\) Poissonian distribution  
\(\rightarrow\) Strength of fluctuations \(\rightarrow\) variance \(\Delta I^2 \equiv \langle (I - \langle I \rangle)^2 \rangle\)  
\(\rightarrow\) Variance depends on frequency \(\rightarrow\) Use noise power:
\[
S(f) = \int_{-\infty}^{+\infty} \left( \langle I(t)I(0) \rangle - \langle I(0) \rangle^2 \right) dt
\]

includes equilibrium fluctuations (white noise)

**1/f noise**

\(\rightarrow\) Dominant at low frequencies  
\(\rightarrow\) Physical nature often unclear  
\(\rightarrow\) Josephson junctions: dominant below about 1 Hz - 1 kHz \(\rightarrow\) Not considered here
3.1.5 The basic junction equation

Kirchhoff’s law:
\[ I = I_S + I_N + I_D + I_F \]

Voltage-phase relation:
\[ \frac{d\varphi}{dt} = \frac{2eV}{\hbar} \]

⇒ Basic equation of a Josephson junction

⇒ \[ I = I_c \sin \varphi + G_N(V)V + C \frac{dV}{dt} + I_F \]

⇒ \[ I = I_c \sin \varphi + G_N(V) \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} + C \frac{\Phi_0}{2\pi} \frac{d^2\varphi}{dt^2} + I_F \]

Nonlinear differential equation with nonlinear coefficients

⇒ Complex behavior, numerical solution

⇒ Use approximations (simple models)
3.2 The resistively and capacitively shunted junction (RCSJ) model

Resistively and Capacitively Shunted Junction (RCSJ) model

Approximation \( G_N(V) \equiv G = R^{-1} = \text{const.} \)

\( R = \text{Junction normal resistance} \)

Differential equation in dimensionless or energy formulation

\[
\left( \frac{\hbar}{2e} \right) C \frac{d^2 \varphi}{dt^2} + \left( \frac{\hbar}{2e} \right) \frac{1}{R} \frac{d \varphi}{dt} + I_c \left[ \sin \varphi - \frac{l}{I_c} + \frac{I_F(t)}{I_c} \right] = 0
\]

\[\equiv i \equiv i_F(t)\]

\[
\Rightarrow \left( \frac{\hbar}{2e} \right)^2 C \frac{d^2 \varphi}{dt^2} + \left( \frac{\hbar}{2e} \right)^2 \frac{1}{R} \frac{d \varphi}{dt} + \frac{d}{d \varphi} \left\{ E_{J0} [1 - \cos \varphi - i \varphi + i_F(t) \varphi] \right\} = 0
\]

Mechanical analog

Gauge invariant phase difference \(\leftrightarrow\) Particle with mass \(M\) and damping \(\eta\) in potential \(U\):

\[
M \frac{d^2 x}{dt^2} + \eta \frac{dx}{dt} + \nabla U = 0
\]

with \(M = \left( \frac{\hbar}{2e} \right)^2 C\) \(\eta = \left( \frac{\hbar}{2e} \right)^2 \frac{1}{R}\)

\[
U = E_{J0} [1 - \cos \varphi - i \varphi + i_F(t) \varphi]
\]

Tilted washboard potential
3.2 The resistively and capacitively shunted junction (RCSJ) model

Finite tunneling probability:
→ Macroscopic quantum tunneling (MQT)

Escape by thermal activation
→ Thermally activated phase slips

Normalized time: \( \tau \equiv \frac{t}{\tau_c} = \frac{t}{2eI_c R/\hbar} \)

Stewart-McCumber parameter: \( \beta_C \equiv \frac{\omega_C^2}{\omega_p^2} = \frac{2e}{\hbar} I_c R_N^2 C \)

Motion of „phase particle“ \( \varphi \) in the tilted washboard potential

Plasma frequency

→ Neglect damping, zero driving and small amplitudes (\( \sin \varphi \approx \varphi \))

Solution: \( \varphi = c \cdot \exp \left( \frac{i \tau}{\sqrt{\beta_C}} \right) = c \cdot \exp \left( \frac{i t}{\sqrt{\beta_C \tau_c}} \right) = c \cdot \exp \left( i \omega_p t \right) \)

Plasma frequency = Oscillation frequency around potential minimum
### 3.2 The resistively and capacitively shunted junction (RCSJ) model

#### The pendulum analog

- Plane mechanical pendulum in uniform gravitational field
- Mass $m$, length $\ell$, deflection angle $\theta$
- Torque $D$ parallel to rotation axis
- Restoring torque: $mg\ell \sin \theta$

**Equation of motion**

$$D = \Theta \ddot{\Theta} + \Gamma \dot{\Theta} + mg\ell \sin \Theta$$

- $\Theta = m\ell^2$ Moment of inertia
- $\Gamma$ Damping constant

**Anallogies**

| $I$ | $D$ |
| $I_c$ | $mg\ell$ |
| $\Phi_0$ | $\Gamma$ |
| $2\pi R$ | $\Theta$ |
| $C\Phi_0$ | $\Theta$ |
| $2\pi$ | $\varphi$ |
| $\varphi$ | $\varphi$ |

For $D = 0$ → Oscillations around equilibrium with

$$\omega = \sqrt{\frac{g}{\ell}} \leftrightarrow \text{Plasma frequency } \omega_p = \sqrt{\frac{2\pi I_c}{\Phi_0 C}}$$

Finite torque ($D > 0$) → Finite $\theta_0$ → Finite, but constant $\varphi_0$ → Zero-voltage state

Large torque (deflection > 90°) → Rotation of the pendulum → Finite-voltage state

Voltage $V$ ↔ Angular velocity of the pendulum
3.2.1 Under- and overdamped Josephson junctions

**Underdamped junction**

\[ \beta_C = \frac{2eI_c R^2 C}{\hbar} \ll 1 \]

Capacitance & resistance large
\[ \rightarrow M \text{ large, } \eta \text{ small} \]
\[ \rightarrow \text{Hysteretic IVC} \]

(Phase particle will retrap immediately at \( I_c \) because of large damping)

**Overdamped junction**

\[ \beta_C = \frac{2eI_c R^2 C}{\hbar} \gg 1 \]

Capacitance & resistance small
\[ \rightarrow M \text{ small, } \eta \text{ large} \]
\[ \rightarrow \text{Non-hysteretic IVC} \]

(Once the phase is moving, the potential has to be tilt back almost into the horizontal position to stop ist motion)
3.3 Response to driving sources

Motivation

„Applied Superconductivity“ → One central question is
→ „How to extract information about the junction experimentally?“

Typical strategy
→ Drive junction with a probe signal and measure response

Examples for probe signals
→ Currents (magnetic fields)
→ Voltages (electric fields)
→ DC or AC
→ Josephson junctions → AC means microwaves!

Prototypical experiment
→ Measure junction IVC
→ Typically done with current bias
3.3.1 Response to a dc current source

Time averaged voltage:

\[ \langle V \rangle = \frac{1}{T} \int_0^T V(t) \, dt = \frac{1}{T} \int_0^T \frac{\hbar}{2e} \frac{d\varphi}{dt} \, dt = \frac{1}{T} \frac{\hbar}{2e} [\varphi(T) - \varphi(0)] = \frac{\Phi_0}{T} \]

Total current must be constant (neglecting the fluctuation source):

\[ I = I_s(t) + I_N(t) + I_D(t) = I_c \sin \varphi(t) + \frac{V(t)}{R} + C \frac{dV(t)}{dt} = \text{const} \]

where: \[ \varphi(t) = \int_0^t \frac{2e}{\hbar} V(t) \, dt \]

\( I > I_c \) → Part of the current must flow as \( I_N \) or \( I_D \)

→ Finite junction voltage \(|V| > 0\)
→ Time varying \( I_s \)
→ \( I_N + I_D \) varies in time
→ Time varying voltage, complicated non-sinusoidal oscillations of \( I_s \),
  Oscillating voltage has to be calculated self-consistently
→ Oscillation frequency \( f = \langle V \rangle / \Phi_0 \)
3.3.1 Response to a dc current source

For $I \gtrapprox I_c$

→ Highly non-sinusoidal oscillations
→ Long oscillation period
→ $\langle V \rangle \propto \frac{1}{T}$ is small

For $I \gg I_c$

→ Almost all current flows as normal current
→ Junction voltage is nearly constant
→ Almost sinusoidal Josephson current oscillations
→ Time averaged Josephson current almost zero
→ Linear/Ohmic IVC

→ Analogy to pendulum
3.3.1 Response to a dc current source

**Strong damping**

\( \beta_c \ll 1 \) & neglecting noise current

\[ i < 1 \rightarrow \text{Only supercurrent, } \varphi = \sin^{-1} i \text{ is a solution, zero junction voltage} \]

\[ i > 1 \rightarrow \text{Finite voltage, temporal evolution of the phase} \]

\[ d\tau = \frac{d\varphi}{i - \sin \varphi} \]

Integration using

\[ \int \frac{dx}{a - \sin x} = \frac{2}{\sqrt{a^2 - 1}} \tan^{-1} \left( \frac{-1 + a \tan(x/2)}{\sqrt{a^2 - 1}} \right) \]

gives

\[ \tau - \tau_0 = \frac{2}{\sqrt{i^2 - 1}} \tan^{-1} \left( \frac{-1 + i \tan(\varphi/2)}{\sqrt{i^2 - 1}} \right) \]

\[ \Rightarrow \varphi(t) = 2 \tan^{-1} \left\{ \sqrt{1 - \frac{1}{i^2}} \tan \left( \frac{t \sqrt{i^2 - 1}}{2\tau_c} \right) + \frac{1}{i} \right\} \]

**Periodic function with period**

\[ T = \frac{2\pi \tau_c}{\sqrt{i^2 - 1}} \]

\( \tan^{-1}(a \tan x + b) \) is \( \pi \)-periodic
3.3.1 Response to a dc current source

with \( \langle V(t) \rangle = \frac{1}{T} \int_{0}^{T} V(t) \, dt = \frac{\Phi_0}{T} \)

and \( \tau_c = \frac{\Phi_0}{2\pi} \frac{1}{I_c R} \)

\( T = \frac{2\pi \tau_c}{\sqrt{i^2 - 1}} \)

We get for \( i > 1 \)

\( \langle V(t) \rangle = I_c R \sqrt{\left( \frac{I}{I_c} \right)^2 - 1} \)
3.3.1 Response to a dc current source

Weak damping

\[ \beta_c \gg 1 \] & neglecting noise current

\[ \omega_{RC} = \frac{1}{R_NC} \] is very small

→ Large \( C \) is effectively shunting oscillating part of junction voltage \( \rightarrow V(t) \approx \bar{V} \)

→ Time evolution of the phase

\[ \varphi(t) = \frac{2e}{\hbar} \bar{V} t + \text{const} \]

→ Almost sinusoidal oscillation of Josephson current

\[ \bar{l}_s(t) = l_c \sin \left( \frac{2e}{\hbar} \bar{V} t + \text{const} \right) \approx 0 \]

→ Down to \( \bar{V} \approx \frac{\hbar \omega_{RC}}{e} \ll (V_c = I_cR_N) \) \( \rightarrow \bar{l} = l_N(\bar{V}) = \frac{\bar{V}}{R} \)

→ Corresponding current \( \ll I_c \rightarrow \text{Hysteretic IVC} \)

Ohmic result valid for \( R_N = \text{const} \).

→ Real junction \( \rightarrow \text{IVC determined by voltage dependence of } R_N = R_N(V) \)
3.3.1 Response to a dc current source

Intermediate damping

\[ \beta_C \approx 1 \]

\[ \rightarrow \text{Numerically solve IVC} \]

\[ \rightarrow \text{General trend} \]

Increasing \( \beta_C \) \( \leftrightarrow \) Increasing hysteresis

Hysteresis characterized by retrapping current \( I_r \)

\[ \rightarrow I_r \propto \text{washboard potential tilt} \]

Energy dissipated in advancing to next minimum = Work done by drive current

\[
\frac{I_R}{I_c} = 4 \frac{1}{\pi \sqrt{\beta_C}}
\]

Analytical calculation possible for \( \beta_C \gg 1 \) (exercise class)
3.3.1 Response to a dc voltage source

Phase evolves linearly in time: \[ \varphi(t) = \frac{2e}{\hbar} V_{dc} t + \text{const} \]

- Josephson current \( I_S \) oscillates sinusoidally
- Time average of \( I_S \) is zero
- \( I_D = 0 \) since \( \frac{dV_{dc}}{dt} = 0 \)
- Total current carried by normal current \( \rightarrow I = \frac{V_{dc}}{R_N} \)

RCSJ model \( \rightarrow \) Ohmic IVC
General case \( R = R_N (V) \) \( \rightarrow \) Nonlinear IVC
3.3.2 Response to ac driving sources

Response to an ac voltage source

Strong damping $\beta_C \ll 1$

$$V(t) = V_{dc} + V_1 \cos \omega_1 t$$

Integrating the voltage-phase relation:

$$\varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0} V_{dc} t + \frac{2\pi}{\Phi_0} \frac{V_1}{\omega_1} \sin \omega_1 t$$

Current-phase relation:

$$I_s(t) = I_c \sin \left\{ \varphi_0 + \frac{2\pi}{\Phi_0} V_{dc} t + \frac{2\pi}{\Phi_0} \frac{V_1}{\omega_1} \sin \omega_1 t \right\}$$

Superposition of linearly increasing and sinusoidally varying phase

→ Supercurrent $I_s(t)$ and ac voltage $V_1$ have different frequencies
→ Origin → Nonlinear current-phase relation
3.3.2 Response to ac driving sources

Some maths for the analysis of the time-dependent Josephson current

Fourier-Bessel series identity:

\[ e^{ib\sin x} = \sum_{n=-\infty}^{+\infty} J_n(b) e^{in\pi x} \]

\[ J_n(b) = n^{th} \text{ order Bessel function of the first kind} \]

and:

\[ \sin(a + b\sin x) = \Im \left\{ e^{i(a+b\sin x)} \right\} \]

\[ e^{i(a+b\sin x)} = \sum_{n=-\infty}^{+\infty} J_n(b) e^{i(a+nx)} = \sum_{n=-\infty}^{+\infty} (-1)^n J_n(b) e^{i(a-nx)} \]

\[ \Rightarrow \sin(a + b\sin x) = \sum_{n=-\infty}^{+\infty} (-1)^n J_n(b) \sin(a - nx) \]

\[ J_{-n}(b) = (-1)^n J_n(b) \]

Imaginary part

Ac driven junction \( \Rightarrow x = \omega_1 t, b = \frac{2\pi}{\Phi_0 \omega_1} \) and \( a = \phi_0 + \omega_{dc} t = \phi_0 + \frac{2\pi}{\Phi_0} V_{dc} t \)

\[ l_s(t) = l_c \sum_{n=-\infty}^{+\infty} (-1)^n J_n \left( \frac{2\pi V_1}{\Phi_0 \omega_1} \right) \sin \left[ (\omega_{dc} - n\omega_1) t + \phi_0 \right] \]

\[ \rightarrow \text{Frequency } \omega_{dc} \text{ couples to multiples of the driving frequency} \]
3.3.2 Response to ac driving sources

Shapiro steps

\[ l_s(t) = l_c \sum_{n=-\infty}^{+\infty} (-1)^n J_n \left( \frac{2\pi V_1}{\Phi_0 \omega_1} \right) \sin \left[ (\omega_{dc} - n\omega_1) t + \varphi_0 \right] \]

→ Ac voltage results in dc supercurrent if \([ (\omega_{dc} - n\omega_1) t + \varphi_0 ]\) is time independent

\[ \omega_{dc} = n\omega_1 \quad \text{or} \quad V_{dc} = V_n = n \frac{\Phi_0}{2\pi} \omega_1 \]

→ Amplitude of average dc current for a specific step number \(n\)

\[ |\langle l_s \rangle_n| = l_c \left| J_n \left( \frac{2\pi V_1}{\Phi_0 \omega_1} \right) \right| \]

\(V_{dc} \neq V_n\)

→ \([ (\omega_{dc} - n\omega_1) t + \varphi_0 ]\) is time dependent

→ Sum of sinusoidally varying terms

→ Time average is zero

→ Vanishing dc component \(\rightarrow \langle I \rangle = \frac{V_{dc}}{R_N} + \langle \frac{V_1}{R_N} \cos \omega_1 t \rangle = \frac{V_{dc}}{R_N} \)
3.3.2 Response to ac driving sources

→ Ohmic dependence with sharp current spikes at $V_{dc} = V_n$

→ Current spike amplitude depends on ac voltage amplitude

→ $n^{th}$ step → Phase locking of the junction to the $n^{th}$ harmonic

$V_n = n \frac{\Phi_0}{2\pi} \omega_1$

$|\langle I_s \rangle_n| = I_c |J_n\left(\frac{2\pi V_1}{\Phi_0 \omega_1}\right)|$

Example: $\omega_1 / 2\pi = 10$ GHz

Constant dc current at $V_{dc} = 0$ and $V_n = n \omega_1 \frac{\Phi_0}{2\pi} \approx n \times 20 \mu V$
3.3.2 Response to ac driving sources

Response to an ac current source

Strong damping $\beta_C \ll 1$ (experimentally relevant)

$\rightarrow$ Kirchhoff’s law (neglecting $I_D$) $\Rightarrow I_c \sin \varphi + \frac{1}{R_N} \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} = I_{dc} + I_1 \sin \omega_1 t$

Difficult to solve $\rightarrow$ Qualitative discussion with washboard potential

$\rightarrow$ Increase $I_{dc}$ at constant $I_1$
$\rightarrow$ Zero-voltage state for $I_{dc} + I_1 \leq I_c$, finite voltage state for $I_{dc} + I_1 > I_c$
$\rightarrow$ Complicated dynamics!

$\rightarrow$ $V_n = n \omega_1 \frac{\Phi_0}{2\pi}$ $\rightarrow$ Motion of phase particle synchronized by ac driving

Simplifying assumption

$\rightarrow$ During each ac cycle the phase particle moves down $n$ minima

$\rightarrow$ Resulting phase change $\dot{\varphi} = n \frac{2\pi}{T} = n \omega_1$

$\rightarrow$ Average dc voltage $\langle V \rangle = n \frac{\Phi_0}{2\pi} \omega_1 \equiv V_n$

Exact analysis

$\rightarrow$ Synchronization of phase dynamics with external ac source for a certain bias current interval $\rightarrow$ Steps
3.3.2 Response to ac driving sources

Experimental IVCs obtained for an underdamped and overdamped Niobium Josephson junction under microwave irradiation

\[ V_n = n \frac{\Phi_0}{2\pi} \omega_1 \]
3.3.4 Photon-assisted tunneling

Superconducting tunnel junction → Highly nonlinear $R(V)$
→ Sharp step at $V_g = \frac{2\Delta}{e}$
→ Use quasiparticle (QP) tunneling current $I_{qp}(V)$
→ Include effect of ac source on QP tunneling

Model of Tien and Gordon:
→ Ac driving shifts levels in electrode up and down
  QP energy: $E_{qp} + eV_1 \cos \omega_1 t$
→ QM phase factor

$$\exp \left( -\frac{i}{\hbar} \int (E_{qp} + eV_1 \cos \omega_1) dt \right)$$
$$= \exp \left( -\frac{i}{\hbar} E_{qp} t \right) \cdot \exp \left( -i \frac{eV_1}{\hbar \omega_1} \sin \omega_1 t \right)$$

Bessel function identity for $V_1$-term → Sum of terms $\mathcal{J}_n \left( \frac{eV_1}{\hbar \omega_1} \right) e^{-in\omega_1 t}$
→ Splitting of qp-levels into many levels $E_{qp} \pm n\hbar \omega_1$ → Modified density of states!

→ Tunneling current $I_{qp}(V) = \sum_{n=-\infty}^{+\infty} \mathcal{J}_n^2 \left( \frac{eV_1}{\hbar \omega_1} \right) I_{qp}^0 (V + n\hbar \omega_1/e)$
→ Sharp increase of the $I_{qp}(V)$ at $V = V_g$ is broken up into many steps of smaller current amplitude at $V_n = V_g \pm \frac{n\hbar \omega_1}{e}$
3.3.4 Photon-assisted tunneling

Example
→ QP IVC of a Nb SIS Josephson junction without & with microwave irradiation
→ Frequency $\omega_1/2\pi = 230$ GHz corresponding to $\hbar \omega_1/e \approx 950 \mu V$

\[ \omega_1/2\pi = 230 \text{ GHz} \]

**QP steps**
→ Appear at $V_n = n \frac{\hbar}{e} \omega_1$
→ Amplitude $J_n \left( \frac{eV_1}{\hbar \omega_1} \right)$
→ Broadened steps (depending on $I_{qp}(V)$)

**Shapiro steps**
→ Appear at $V_n = n \frac{\hbar}{2e} \omega_1$
→ Amplitude $J_n \left( \frac{2eV_1}{\hbar \omega_1} \right)$
→ Sharp steps
3.4 Additional topic: Effect of thermal fluctuations

Thermal fluctuations with correlation function:
\[ \langle I_F(t)I_F(t + \tau) \rangle = \frac{2k_B T}{R_N} \delta(\tau) \]

\[ S(f) = 4k_B T/R_N \]

Larger fluctuations
→ Increase probability for escape out of potential well
→ Escape at rates \( \Gamma_{n+1} \)
  → Escape to next minimum
  → Phase change of \( 2\pi \)
→ \( l > 0 \) → \( \Gamma_{n+1} > \Gamma_{n-1} \) → \( \frac{d\varphi}{dt} > 0 \)

Langevin equation for RCSJ model
\[ l = l_c \sin \varphi + \frac{1}{R_N} \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} + C \frac{\Phi_0}{2\pi} \frac{d^2\varphi}{dt^2} + I_F \]
→ Equivalent to Fokker-Planck equation:
\[ \frac{1}{\omega_c} \frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial \varphi} (\sigma \nu) + \frac{1}{\beta_C} \frac{\partial}{\partial \nu} (\sigma [f(\varphi) - \nu]) = \frac{\gamma}{\beta_C^2} \frac{\partial^2 \sigma}{\partial \nu^2} \]

Normalized force
\[ f(\varphi) = -\frac{1}{E_{J0}} \frac{\partial U(\varphi)}{\partial \varphi} = \frac{l}{l_c} - \sin \varphi \]

Normalized momentum
\[ \nu = \frac{d\varphi/dt}{\omega_c} = \frac{V}{l_c R_N} \]
3.4 Additional topic: Effect of thermal fluctuations

\[ \sigma(v, \varphi, t) \rightarrow \text{Probability density of finding system at } (v, \varphi) \text{ at time } t \]

\[ \langle X \rangle(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma(\varphi, v, t)X(\varphi, v, t)d\varphi dv \]

**Small fluctuations**

\[ \rightarrow \text{Static solution } \left( \frac{d\sigma}{dt} = 0 \right) \quad \sigma(v, t) = \mathcal{F}^{-1} \exp \left( -\frac{G(\varphi, \sigma)}{k_B T} \right) \]

with: \[ \mathcal{F} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left( -\frac{G(\varphi, \sigma)}{k_B T} \right) d\varphi dv \]

\[ \rightarrow \text{Boltzmann distribution } (G = E - Fx \text{ is total energy, } E \text{ is free energy}) \]

\[ \rightarrow \text{Constant probability to find system in } n^{\text{th}} \text{ metastable state} \quad p = \int_{-\infty}^{+\infty} d\varphi \int_{\varphi \approx \varphi_n}^{+\infty} \sigma(\varphi, v) d\varphi \]
3.4 Additional topic: Effect of thermal fluctuations

Large fluctuations

\[ p \text{ can change in time} \rightarrow \frac{dp}{dt} = (\Gamma_{n+1} - \Gamma_{n-1})p \]

\[ \text{Amount of phase slippage} \]

for \( \Gamma_{n+1} \gg \Gamma_{n-1} \) and \( \frac{\omega_A}{\Gamma_{n+1}} \gg 1 \) \rightarrow \( \Gamma_{n+1} = \frac{\omega_A}{2\pi} \exp \left( -\frac{U_0}{k_B T} \right) \)

\( \omega_A = \) Attempt frequency

Attempt frequency \( \omega_A \)

\[ \omega_A = \omega_0 = \omega_p (1 - i^2)^{1/4} \quad \text{for} \quad \omega_c \tau \gg 1, \quad \text{(underdamped junction)} \]

\[ \omega_A = \tau^{-1} = \omega_c (1 - i^2)^{1/2} \quad \text{for} \quad \omega_c \tau \ll 1 \quad \text{(overdamped junction)} \]

Weak damping (\( \beta_c = \omega_c \tau_{RC} \gg 1 \))

\( \rightarrow I = 0 \rightarrow \omega_A = \omega_p \) (Oscillation frequency in the potential well)

\( \rightarrow I \ll I_c \rightarrow \omega_A \gg \omega_p \)

Strong damping (\( \beta_c = \omega_c \tau_{RC} \ll 1 \))

\( \rightarrow \omega_p \rightarrow \omega_c \) (Frequency of an overdamped oscillator)
3.4.1 Underdamped junctions: Critical current reduction by premature switching

For $E_J \gg k_B T \rightarrow$ Small escape probability $\propto \exp \left( - \frac{U_0(I)}{k_B T} \right)$ at each attempt

Barrier height: $U_0(I) \simeq \frac{E_J}{2} \left( 1 - \frac{I}{I_c} \right)^{3/2}$

$\rightarrow \frac{2E_J}{2}$ for $I = 0$
$\rightarrow 0$ for $I \rightarrow I_c$

Escape probability $\rightarrow \omega_A/2\pi$ for $I \rightarrow I_c$
After escape $\rightarrow$ Junction switches to $IR_N$

Experiment

$\rightarrow$ Measure distribution of escape current $I_M$
$\rightarrow$ Width $\delta I$ and mean reduction $\langle \Delta I_c \rangle = I_c - \langle I_M \rangle$
$\rightarrow$ Use approximation for $U_0(I)$ and escape rate
$\omega_A/2\pi \exp \left( - \frac{U_0(I)}{k_B T} \right)$

$\langle \Delta I_c \rangle = I_c - \langle I_M \rangle \simeq I_c \left( \frac{k_B T}{2E_J} \ln \left( \frac{\omega_p \Delta t}{2\pi} \right) \right)^{2/3}$

$\rightarrow$ Considerable reduction of $I_c$ when $k_B T > 0.05 E_J$

$\rightarrow$ Provides experimental information on real or effective temperature!
3.4.2 Overdamped junctions: The Ambegaokar-Halperin theory

Calculate voltage $\langle V \rangle$ induced by thermally activated phase slips as a function of current

Important parameter:

$$\gamma_0(T) = \frac{2E_{j0}(T)}{k_B T} = \frac{\Phi_0 I_c(T)}{\pi k_B T}$$
3.4.2 Overdamped junctions: The Ambegaokar-Halperin theory

**Amegaokar-Halperin theory**

Finite amount of phase slippage
- Nonvanishing voltage for \( I \to 0 \)
- Phase slip resistance for strong damping \((\beta_C \ll 1)\), for \( U_0 = 2E_{J0} \):

\[
R_p = \lim_{I \to 0} \frac{\langle V \rangle}{I} = R_N \left\{ \mathcal{I}_0 \left[ \frac{\gamma_0(T)}{2} \right] \right\}^{-2} \\
\gamma_0(T) = \frac{2E_{J0}(T)}{k_B T} = \frac{\Phi_0 I_C(T)}{\pi k_B T}
\]

Modified Bessel function

\[
\frac{E_{J0}}{k_B T} \gg 1 \Rightarrow \text{Approximate Bessel function} \Rightarrow \mathcal{I}_0(x) = e^x / 2\pi \sqrt{x}
\]

\[
\frac{R_p(T)}{R_N} \propto E_{J0} \exp \left( -\frac{2E_{J0}}{k_B T} \right)
\]

\[
\langle \dot{\phi} \rangle \propto \frac{2eI_C R_N}{\hbar} \exp \left( -\frac{2E_{J0}}{k_B T} \right) = \omega_c \exp \left( -\frac{2E_{J0}}{k_B T} \right)
\]

**Attempt frequency** is characteristic frequency \( \omega_c \)

Plasma frequency has to be replaced by frequency of overdamped oscillator:

\[
\omega_A = \omega_p \sqrt{\beta_C} = \omega_p \sqrt{\omega_c R_N C} = \omega_c
\]

Washboard potential \( \Rightarrow \) Phase diffuses over barrier \( \Rightarrow \) Activated nonlinear resistance
3.4.2 Overdamped junctions: The Ambegaokar-Halperin theory

Example: $\text{YBa}_2\text{Cu}_3\text{O}_7$ grain boundary Josephson junctions
→ Strong effect of thermal fluctuations due to high operation temperature

- $R.\text{ Gross et al.}, \quad \text{Phys. Rev. Lett. 64, 228 (1990)}$
- $\quad \text{Nature 322, 818 (1988)}$
Overdamped YBa$_2$Cu$_3$O$_7$ grain boundary Josephson junction

Overdamped junctions: The Ambegaokar-Halperin theory

\[ \frac{2E_{J0}(0)}{k_B T} = 600 \]

\[ \frac{2E_{J0}(0)}{k_B T} = 1400 \]

Determination of \( I_c(T) \) close to \( T_c \)

3.5 Voltage state of extended Josephson junctions

So far
→ Junction treated as lumped element circuit element
→ Spatial extension neglected

Spatially extended junctions
→ Specific geometry as as in Chapter 2
  → Insulating barrier in $yz$-plane
  → In-plane $B$ field in $y$-direction
  → Thick electrodes $\gg \lambda_{L1,2}$
  → Magnetic thickness $t_B = d + \lambda_{L1} + \lambda_{L2}$
  → Bias current in $x$-direction
→ Phase gradient along $z$-direction
  → $\frac{\partial \varphi(z,t)}{\partial z} = \frac{2\pi}{\Phi_0} t_B B_y(z,t)$

Expected effects
→ Voltage state $\rightarrow$ $E$-field and time-dependence become important
→ Short junction and long junction case
3.5.1 Negligible Screening Effects

Neglect self-fields (short junctions)

$\rightarrow B = B^{\text{ex}}$

$\rightarrow$ Junction voltage $V = $ Applied voltage $V_0$

$\rightarrow$ Gauge invariant phase difference:

$$ \frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} V_0 = \omega_0 $$

$$ \frac{\partial \varphi(z, t)}{\partial z} = \frac{2\pi}{\Phi_0} t_B B_y(z, t) $$

$\Rightarrow \varphi(z, t) = \varphi_0 + \omega_0 t + \frac{2\pi}{\Phi_0} B_y t_B \cdot z = \varphi_0 + \omega_0 t + k \cdot z$

$\Rightarrow J_s(z, t) = J_c \sin(\omega_0 t + k \cdot z + \varphi_0)$

$\rightarrow$ Josephson vortices moving in $z$-direction with velocity

$$ v_z = \frac{\omega_0}{k} = \frac{V_0}{B_y t_B} $$

$\begin{array}{l}
\text{Graph 1:}
\begin{array}{cc}
\varphi / 2\pi & z / L
\end{array}
\text{Graph 2:}
\begin{array}{cc}
J_s(z) \text{ (arb. units)} & z / L
\end{array}
\end{array}$
3.5.2 The time dependent Sine-Gordon equation

Long junctions ($L \gg \lambda_J$)

- Effect of Josephson currents has to be taken into account
- Magnetic flux density = External + Self-generated field

with $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \varepsilon_0 \mathbf{E}$:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

in contrast to static case, now $\partial \mathbf{E}/\partial t \neq 0$

Consider 1D junction extending in $z$-direction, $B = B_y$, current flow in $x$-direction

$$\frac{\partial B_y(z, t)}{\partial z} = -\mu_0 J_x(z, t) - \varepsilon_0 \mu_0 \frac{\partial E_x(z, t)}{\partial t}$$

$$\Rightarrow \frac{\partial^2 \varphi(z, t)}{\partial z^2} = -\frac{2\pi}{\Phi_0} t_B \left\{ \mu_0 J_x(z, t) + \varepsilon_0 \mu_0 \frac{\partial E_x(z, t)}{\partial t} \right\}$$

with $E_x = -V/d$, $J_x = -J_c \sin \varphi$ and $\partial \varphi/\partial t = 2\pi V/\Phi_0$:

$$\frac{\partial^2 \varphi(z, t)}{\partial z^2} = \frac{2\pi t_B \mu_0 J_c}{\Phi_0} \sin \varphi(z, t) + \frac{\varepsilon_0 \mu_0 t_B}{d} \frac{\partial^2 \varphi(z, t)}{\partial t^2}$$

$\lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}}$

(Josephson penetration depth)

$\overline{c} = \sqrt{\frac{d}{\varepsilon_0 \mu_0 t_B}}$

(propagation velocity)
### 3.5.2 The time dependent Sine-Gordon equation

\[
\frac{\partial^2 \varphi(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi(z, t)}{\partial t^2} - \frac{1}{\lambda_j^2} \sin \varphi(z, t) = 0
\]

with the Swihart velocity

\[
\bar{c} = \sqrt{\frac{d}{\varepsilon \mu_0 \mu_0 t_B}} = \frac{1}{\sqrt{\varepsilon \mu_0}} \sqrt{\frac{d}{\varepsilon (2\lambda_L + d)}} = c \sqrt{\frac{1}{\varepsilon (1 + 2\lambda_L/d)}}
\]

\(\bar{c}\) = velocity of TEM mode in the junction transmission line

Example: \(\varepsilon \approx 5 - 10, \frac{2\lambda_L}{d} \approx 50 - 100 \Rightarrow \bar{c} \approx 0.1c\)

\(\Rightarrow\) Reduced wavelength

\(\Rightarrow\) For \(f = 10\ GHz\) \(\Rightarrow\) Free space: 3 cm, in junction: 1 mm

Other form of time-dependent Sine-Gordon equation

\[
\lambda_j^2 \frac{\partial^2 \varphi(z, t)}{\partial z^2} - \frac{4\pi^2}{\omega_p^2} \frac{\partial^2 \varphi(z, t)}{\partial t^2} - \sin \varphi(z, t) = 0
\]

\[
\omega_p^2 = 2e l_c / \hbar C \quad C/A_i = \varepsilon \varepsilon_0 / d \quad l_c/A_i = J_c \quad c^2 = 1/\varepsilon_0 \mu_0 \quad \Rightarrow \omega_p / 2\pi = \bar{c} / \lambda_j
\]
3.5.2 The time dependent Sine-Gordon equation

Time-dependent Sine-Gordon equation:

\[ \lambda_j^2 \frac{\partial^2 \varphi(z, t)}{\partial z^2} - \frac{4\pi^2}{\omega_p^2} \frac{\partial^2 \varphi(z, t)}{\partial t^2} - \sin \varphi(z, t) = 0 \]

**Mechanical analogue**

→ Chain of mechanical pendula attached to a twistable rubber ribbon

→ Restoring torque \( \lambda_j^2 \frac{\partial^2 \varphi}{\partial z^2} \)

→ Short junction w/o magnetic field
  → \( \frac{\partial^2 \varphi}{\partial z^2} = 0 \)
  → Rigid connection of pendula
  → Corresponds to single pendulum
3.5.3 Solutions of the time dependent SG equation

Simple case

→ 1D junction \((W \ll \lambda_J)\), short and long junctions

Short junctions \((L \ll \lambda_J)\) @ low damping

→ Neglect z-variation of \(\varphi\)

\[
\frac{\partial^2 \varphi(z, t)}{\partial t^2} + \frac{\omega_p^2}{4\pi^2} \sin \varphi(z, t) = 0
\]

→ Equivalent to RCSJ model for \(G_N = 0, I = 0\)

Small amplitudes → Plasma oscillations

(Oscillation of \(\varphi\) around minimum of washboard potential)

Long junctions \((L \gg \lambda_J)\)

→ Solution for infinitely long junction → Soliton or fluxon

\[\varphi(z, t) = 4 \arctan \left\{ \exp \left( \pm \frac{z - z_0}{\lambda_J} - \frac{V_z}{c} t \right) \right\} \sqrt{1 - \left( \frac{V_z}{c} \right)^2} \]

\(\varphi = \pi\) at \(z = z_0 + V_z t\)

goes from 0 to \(2\pi\) for \(-\infty \to z \to \infty\)

→ Fluxon (antifluxon: \(\infty \to z \to -\infty\)
3.5.3 Solutions of the time dependent SG equation

\[ \varphi = \pi \text{ at } z = z_0 + v_z t \]

goes from 0 to \(2\pi\) for \(-\infty \rightarrow z \rightarrow \infty\)

\[ \rightarrow \text{Fluxon (antifluxon: } \infty \rightarrow z \rightarrow -\infty) \]

Pendulum analog

\[ \rightarrow \text{Local } 360^\circ \text{ twist of rubber ribbon} \]

Applied current

\[ \rightarrow \text{Lorentz force } \rightarrow \text{Motion of phase twist (fluxon)} \]

Fluxon as particle \[ \rightarrow \text{Lorentz contraction for } v_z \rightarrow \overline{c} \]

Local change of phase difference \[ \rightarrow \text{Voltage} \]

\[ \rightarrow \text{Moving fluxon } = \text{Voltage pulse} \]

Other solutions: Fluxon-fluxon collisions, ...
3.5.3 Solutions of the time dependent SG equation

**Josephson plasma waves**

**Linearized Sine-Gordon equation**

\[ \varphi(z, t) = \varphi_0(z) + \varphi_1(z, t) \]

- \( \varphi_1 = \) Small deviation
- \( \to \) Approximation \( \sin \varphi \approx \sin \varphi_0 + \varphi_1 \cos \varphi_0 \)

Substitution (keeping only linear terms):

\[
\frac{\partial^2 \varphi_0}{\partial z^2} + \frac{\partial^2 \varphi_1(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi_1(z, t)}{\partial t^2} - \frac{1}{\lambda_j^2} \sin \varphi_0 - \frac{1}{\lambda_j^2} \cos \varphi_0 \varphi_1(z, t) = 0
\]

- \( \varphi_0 \) solves time independent SGE \( \frac{\partial^2 \varphi_0}{\partial z^2} = \lambda_j^{-2} \sin \varphi_0 \)

\[
\frac{\partial^2 \varphi_1(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi_1(z, t)}{\partial t^2} - \frac{1}{\lambda_j^2} \cos \varphi_0 \varphi_1(z, t) = 0
\]

- \( \varphi_0 \) slowly varying \( \to \) \( \varphi_0 \approx \text{const.} \)
3.5.3 Solutions of the time dependent SG equation

Solution: \( \varphi_1(z, t) = \exp (-i[kz - \omega t]) \)

(small amplitude plasma waves)

\[
\omega^2 = \bar{c}^2 k^2 + \omega_{p,J}^2
\]

Dispersion relation \( \omega(k) : \)

Josephson plasma frequency

\[
\frac{\omega_{p,J}^2}{4\pi^2} = \frac{\bar{c}^2}{\lambda_J^2} \cos \varphi_0 = \frac{\omega_p^2}{4\pi^2} \cos \varphi_0
\]

\( \omega < \omega_{p,J} \)
→ Wave vector \( k \) imaginary → No propagating solution

\( \omega > \omega_{p,J} \)
→ Mode propagation
→ Pendulum analogue → Deflect one pendulum → Relax → Wave like excitation

\( \omega = \omega_{p,J} \)
→ Infinite wavelength Josephson plasma wave
→ Analogy to plasma frequency in a metal
→ Typically junctions \( \omega_{p,J} \approx 10 \text{ GHz} \)

Plane waves

For very large \( \lambda_J \) or very small \( I \)
→ Neglect \( \frac{\sin \varphi}{\lambda_J^2} \) term → Linear wave equation → Plane waves with velocity \( \bar{c} \)
3.5.4 Resonance phenomena

Interaction of fluxons or plasma waves with oscillating Josephson current

→ Rich variety of interesting resonance phenomena
→ Require presence of $B^{\text{ex}}$
→ Steps in IVC (junction upconverts dc drive)

**Flux-flow steps and Eck peak**

For $B^{\text{ex}} > 0$

→ Spatially modulated Josephson current density moves at $v_z = V / B_y t_B$
→ Josephson current can excite Josephson plasma waves
→ On resonance, em waves couple strongly to Josephson current if $\bar{c} = v_z$

Corresponding junction voltage:

$$V_{\text{Eck}} = \bar{c} B_y t_B = \sqrt{\frac{d}{\varepsilon_0 \mu_0 t_B}} B_y t_B = \frac{\omega_p \lambda_J}{2\pi L} B_y t_B L = \frac{\omega_p \lambda_J}{2\pi L} \frac{\Phi}{\Phi_0}$$

→ Eck peak at frequency:

$$\omega_{\text{Eck}} = \frac{2e}{\hbar} V_{\text{Eck}} = \omega_p \frac{\lambda_J}{L} \frac{\Phi}{\Phi_0}$$

$\bar{c} = \frac{\omega_p \lambda_J}{2\pi \Phi}$

$\Phi = B_y t_B L$
3.5.4 Resonance phenomena

→ Traveling current wave only excites traveling em wave of same direction
  → Low damping, short junctions  → Em wave is reflected at open end
  → Eck peak only observed in long junctions at medium damping when the backward wave is damped

Alternative point of view

→ Lorentz force  → Josephson vortices move at \( v_Z = \frac{V}{B_y t_B} \)
→ Increase driving force  → Increase \( v_Z \)
→ Maximum possible speed is \( v_Z = \bar{c} \)
→ Further increase of \( I \) does not increase \( V \)
→ Flux flow step in IVC
→ \( V_{ffs} = \bar{c} B_y t_B = \bar{c} \frac{\Phi}{L} = \frac{\omega_p}{2\pi} \frac{\lambda_j}{L} \Phi_0 \frac{\Phi}{\Phi_0} \)
→ Corresponds to Eck voltage
3.5.4 Resonance phenomena

**Fiske steps**

Standing em waves in junction “cavity” at \( \omega_n = 2\pi f_n = 2\pi \frac{\bar{c}}{2L} n = \frac{\pi \bar{c}}{L} n \)

→ Fiske steps at voltages

\[
V_n = \frac{\hbar}{2e} \omega_n = \Phi_0 \frac{\bar{c}}{2L} n = \frac{\omega_p \lambda_j}{2\pi} \frac{\Phi_0}{L} \frac{n}{2}
\]

Interpretation

→ Wave length of Josephson current density is \( \frac{2\pi}{\kappa} \)

→ Resonance condition \( L = \frac{\bar{c}}{2f_n} n = \frac{\lambda}{2} n \Rightarrow kL = n\pi \) or \( \Phi = n \frac{\Phi_0}{2} \)

where maximum Josephson current of short junction vanishes

→ Standing wave pattern of em wave and Josephson current match

→ Steps in IVC

Influence of dissipation

→ Damping of standing wave pattern by dissipative effects
  → Broadening of Fiske steps
  → Observation only for small and medium damping
3.5.4 Resonance phenomena

Fiske steps at small damping and/or small magnetic field

Eck peak at medium damping and/or medium magnetic field

For $V \neq V_{\text{Eck}}$ and $V \neq V_n \rightarrow \langle I_s \rangle = \langle I_c \sin(\omega_0 t + kz + \varphi_0) \rangle \approx 0 \rightarrow I = I_N(V) = V/R_N(V)$
### 3.5.4 Resonance phenomena

#### Zero field steps

- **Motion of trapped flux** due to Lorentz force (w/o magnetic field)
- **Junction of length** $L$, moving back and forth
- **Phase change of** $4\pi$ in period $T = \frac{2L}{v_z}$
- **At large bias currents** ($v_z \to \bar{c}$)

\[
V_{zfs} = \frac{\dot{\phi}}{2e} = \frac{4\pi}{T} \frac{\hbar}{2e} = \frac{4\pi}{2L/\bar{c}} \frac{\hbar}{2e} = \frac{h}{2e} \frac{\bar{c}}{L} = \frac{\omega_p \lambda_J}{\pi L} \Phi_0
\]

For $n$ fluxons

- $V_{n,zfs} = nV_{zfs}$
- $V_{n,zfs} = 2 \times$ Fiske voltage $V_n$ (fluxon has to move back and forth)
- $V_{ffs} = V_{n,zfs}$ for $\Phi = n\Phi_0$ (introduce $n$ fluxons = generate $n$ flux quanta)

**Example:**

IVCs of **annular Nb/insulator/Pb**

**Josephson junction** containing a different number of trapped fluxons

![Graph showing IV curves](image-url)
Voltage state: (Josephson + normal + displacement + fluctuation) current = total current

\[ I = I_c \sin \varphi + G_N(V) \frac{dV}{dt} + I_F \]

\[ \frac{d\varphi}{dt} = \frac{2eV}{\hbar} \]

\[ I = I_c \sin \varphi + G_N(V) \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} + C \frac{\Phi_0}{2\pi} \frac{d^2\varphi}{dt^2} + I_F \]

Equation of motion for phase difference \( \varphi \):

RCSJ-model \( (G_N(V) = \text{const.}) \)

\[ \beta C \frac{d^2\varphi}{d\tau^2} + \frac{d\varphi}{d\tau} + \sin \varphi - i - i_F(\tau) = 0 \]

Motion of phase particle in the tilted washboard potential \( U = E_{J0}[1 - \cos \varphi - (I/I_c)\varphi] \)

Equivalent LCR resonator, characteristic frequencies:

\[ \omega_p = \sqrt{\frac{1}{L_cC}} = \sqrt{\frac{2eI_c}{\hbar C}} \quad \omega_c = \frac{R}{L_c} = \frac{2eI_cR}{\hbar} \quad \omega_{RC} = \frac{1}{RC} \]

Quality factor:

\[ Q^2 = \beta C \equiv \frac{2e}{\hbar} I_c R^2 C \quad \beta C = \text{Stewart-McCumber parameter} \]
Summary (Voltage state of short junctions)

IVC for strong damping and $\beta_c \ll 1$

$$\langle V(t) \rangle = I_c R \sqrt{\left( \frac{l}{l_c} \right)^2 - 1} \quad \text{for} \quad \frac{l}{l_c} > 1$$

Driving with $V(t) = V_{dc} + V_1 \cos \omega_1 t$

$\rightarrow$ Shapiro steps at $V_n = n \frac{\Phi_0}{2\pi} \omega_1$

with amplitudes $|\langle I_s \rangle_n| = I_c \left| J_n \left( \frac{2\pi V_1}{\Phi_0 \omega_1} \right) \right|$

$\rightarrow$ Photon assisted tunneling

Voltage steps at $V_n = n \frac{\Phi_0}{\pi} \omega_1$ due to nonlinear QP resistance

Effect of thermal fluctuations

$\rightarrow$ Phase-slips at rate $\Gamma_{n+1} = \frac{\omega_A}{2\pi} \exp \left( - \frac{U_0}{k_B T} \right)$

$\rightarrow$ Finite phase-slip resistance $R_p$ even below $I_c$

$\rightarrow$ Premature switching
Summary

• voltage state of extended junctions w/o self-field:

\[ J_s(z, t) = J_c \sin(\omega_0 t + k \cdot z + \varphi_0) \]

• with self-field: time dependent Sine-Gordon equation

\[ \frac{\partial^2 \varphi(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi(z, t)}{\partial t^2} - \frac{1}{\lambda_j^2} \sin \varphi(z, t) = 0 \]

\[ \overline{c} = \sqrt{\frac{d}{\varepsilon \varepsilon_0 \mu_0 t_B}} = c \sqrt{\frac{1}{\varepsilon (1 + 2\lambda_L/d)}} \]

\[ \lambda_j \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \]

characteristic velocity of TEM mode in the junction transmission line

Prominent solutions: plasma oscillations and solitons

**nonlinear interactions** of these excitations with Josephson current:

\[ \rightarrow \text{flux-flow steps, Fiske steps, zero-field steps} \]