Introduction to Chapter 6
(from Chapter 3 of the lecture notes)

Quantum treatment of JJ
3.6 Full Quantum Treatment of Josephson Junctions
Secondary Quantum Macroscopic Effects

Classical treatment of Josephson junctions (so far)

Phase $\varphi$ and charge $Q = CV \propto \frac{dq}{dt} \rightarrow$ Purely classical variables

$(Q, \varphi)$ are assumed to be measurable simultaneously

Dynamics $\rightarrow$ Tilted washboard potential, rotating pendulum

Classical energies:

$\rightarrow$ Potential energy $U(\varphi)$
  (Josephson coupling energy / Josephson inductance)

$\rightarrow$ Kinetic energy $K(\varphi)$

  (Charging energy via $\frac{1}{2}CV^2 = \frac{Q^2}{2C} \propto \left(\frac{d\varphi}{dt}\right)^2$ / junction capacitance)

Current-phase & voltage-phase relation from macroscopic quantum model

$\rightarrow$ Quantum origin

$\rightarrow$ Primary macroscopic quantum effects

Second quantization

$\rightarrow$ Treat $(Q, \varphi)$ as quantum variables (commutation relations, uncertainty)

$\rightarrow$ Secondary macroscopic quantum effects
3.6.1 Quantum Consequences of the small Junction Capacitance

**Validity of classical treatment**

Consider an isolated, low-damping junction, \( I = 0 \)

- Cosine potential, depth \( 2E_{J0} \)
- Close to potential minimum

→ Harmonic oscillator
Frequency \( \omega_p \), level spacing \( \hbar \omega_p \)

Vacuum energy \( \frac{\hbar \omega_p}{2} \)

\[ \hbar \omega_p = \sqrt{8E_{J0}E_C} \]
\[ E_C = \frac{e^2}{2C} \]

→ Classical treatment valid for \( \frac{E_{J0}}{\hbar \omega_p} \approx \left( \frac{E_{J0}}{E_C} \right)^{1/2} \gg 1 \) (Level spacing \( \ll \) Potential depth)

\[ E_C \propto \frac{1}{C} \propto \frac{1}{A} \]
\[ E_{J0} \propto I_c \propto A \]

→ Enter quantum regime by decreasing junction area \( A \)
### 3.6.1 Quantum Consequences of the small Junction Capacitance

**Parameters for the quantum regime**

#### Example 1
Area $A = 10 \, \mu m^2$, Tunnel barrier $d = 1 \, nm$, $\varepsilon = 10$,
$J_c = 100 \frac{A}{cm^2}$
$\Rightarrow E_{J0} = 3 \times 10^{-21} \, J$
$\Rightarrow E_{J0}/h = 4500 \, GHz$
$C = \frac{\varepsilon \varepsilon_0 A}{d} = 0.9 \, pF$
$\Rightarrow E_{C} = 2 \times 10^{-26} \, J$
$\Rightarrow \frac{E_{C}}{h} = 30 \, MHz$
$\Rightarrow$ Classical junction

#### Example 2
Area $A = 0.02 \, \mu m^2$
$\Rightarrow C \approx 1 \, fF \Rightarrow E_{C} \approx E_{J0}$
$\Rightarrow$ Quantum junction

$\Rightarrow$ We also need $T \ll 500 \, mK$ for $k_B T \ll E_{J0}, E_{C}$!
3.6.1 Quantum Consequences of the small Junction Capacitance

**Hamiltonian of a strongly underdamped junction** (with $\frac{d\varphi}{dt} \neq 0$)

Kinetic energy:

$$K = \frac{1}{2} CV^2 = \frac{Q^2}{2C} = \frac{1}{2} C \left( \frac{\hbar}{2e} \right)^2 \dot{\varphi}^2 = \frac{1}{2} E_{j0} \frac{\dot{\varphi}^2}{\omega_p^2}$$

$\Rightarrow$ Energy due to extra charge $Q$ on one junction electrode due to $V$

Total energy:

$$E = K + U = E_{j0} \left( 1 - \cos \varphi + \frac{1}{2} \frac{\dot{\varphi}^2}{\omega_p^2} \right)$$

$$U(\varphi) \propto 1 - \cos \varphi \quad \Rightarrow \text{Potential energy}$$

$$K(\dot{\varphi}) \propto \dot{\varphi}^2 \quad \Rightarrow \text{Kinetic energy}$$

Consider $E(\varphi, \dot{\varphi})$ as junction Hamiltonian, rewrite kinetic energy

$$K = \frac{Q^2}{2C} = \frac{1}{2} \frac{1}{(\hbar/2e)^2 C} \left( \frac{\hbar Q}{2e} \right)^2$$

$$\Rightarrow p = \left( \frac{\hbar}{2e} \right) Q$$

$\Rightarrow$ Position coordinate associated to phase $\varphi$, momentum associated to charge $Q$
3.6.1 Quantum Consequences of the small Junction Capacitance

**Canonical quantization** (operator replacement)

$$\frac{\hbar}{2e} Q \rightarrow -i\hbar \frac{\partial}{\partial \varphi}$$

with $N = \frac{Q}{2e} \rightarrow$ # of Cooper pairs

$$Q = -i2e \frac{\partial}{\partial \varphi}$$

we get the Hamiltonian

$$\mathcal{H} = \frac{Q^2}{2C} + E_{J0}(1 - \cos \varphi) = -\frac{(2e)^2}{2C} \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi)$$

$$\mathcal{H} = -4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi)$$

→ Describes only Cooper pairs

$$E_C = \frac{e^2}{2C}$$ nevertheless defined as charging energy for a single electron charge

Commutation rules for the operators

$$[\varphi, Q] = i2e ; \quad [\varphi, N] = i \quad \text{or} \quad [\varphi, \frac{\hbar}{2e} Q] = i\hbar$$

$$N \equiv \frac{Q}{2e} \rightarrow \text{Deviation of # of CP in electrodes from equilibrium}$$

Heisenberg uncertainty relation

$$\Delta N \cdot \Delta \varphi \geq 1$$
3.6.1 Quantum Consequences of the small Junction Capacitance

Hamiltonian in the flux basis \((\phi = \frac{\hbar}{2e} \varphi = \frac{\Phi_0}{2\pi} \varphi)\)

\[
\mathcal{H} = \frac{Q^2}{2C} + E_{J0} \left(1 - \cos 2\pi \frac{\phi}{\Phi_0}\right) = -\frac{(2e)^2}{2C} \frac{\hbar^2}{(2e)^2} \frac{\partial^2}{\partial \phi^2} + E_{J0} \left(1 - \cos 2\pi \frac{\phi}{\Phi_0}\right) \\
= -\frac{\hbar^2}{2C} \frac{\partial^2}{\partial \phi^2} + E_{J0} \left(1 - \cos 2\pi \frac{\phi}{\Phi_0}\right)
\]

Commutator \([\phi, Q] = i\hbar\)

→ \(\phi\) and \(Q\) are **canonically conjugate** (analogous to \(x\) and \(p\))

→ Circuit variables are now quantized

→ **Superconducting quantum circuits**
3.6.2 Limiting Cases: The Phase and Charge Regime

The phase regime

\[ \hbar \omega_p \ll E_{J0}, E_C \ll E_{J0} \rightarrow \text{Phase } \varphi \text{ is a good quantum number!} \]

Lowest energy levels localized near bottom of potential wells at \( \varphi_n = 2\pi n \)

Taylor series for \( U(\varphi) \rightarrow \text{Harmonic oscillator,} \)

Frequency \( \omega_p \), eigenenergies \( E_n = \hbar \omega_p \left( n + \frac{1}{2} \right) \)

Ground state: narrowly peaked wave function at \( \varphi = \varphi_n \)

Large fluctuations of \( Q \) on electrodes since \( \Delta Q \cdot \Delta \varphi \geq 2e \)

(smaller \( E_C \rightarrow \) pairs can easily fluctuate, large \( \Delta Q \))

Small phase fluctuations \( \Delta \varphi \)

Negligible \( \Delta \varphi \Rightarrow \) classical treatment of phase dynamics is good approximation
3.6.2 Limiting Cases: The Phase and Charge Regime

The phase regime

Hamiltonian
\[ \mathcal{H} = -4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi) \]

define \( a = (E - E_{J0})/E_C, \) \( b = E_{J0}/2E_C \) and \( z = \varphi / 2 \)

\[ \to \text{Mathieu equation} \quad \frac{\partial^2 \psi}{\partial z^2} + (a + 2b \cos 2z) \psi = 0 \]

General solution
\[ \psi(\varphi) = \sum_q c_q \psi_q \]

Bloch waves
\[ \psi_q(\varphi) = u_q(\varphi) \exp(iq\varphi) \quad \text{with} \quad u_q(\varphi) = u_q(\varphi + 2\pi) \]

Charge/pair number variable \( q \) is continuous (charge on capacitor!)
\[ \to \Psi(\varphi) \text{ is not } 2\pi\text{-periodic} \]

known from periodic potential problem in solid state physics
\[ \to \text{Energy bands} \]
3.6.2 Limiting Cases: The Phase and Charge Regime

The phase regime

1D problem → Numerical solution straightforward
Variational approach for approximate ground state

Trial function for $E_C \ll E_J$ \quad $\psi(\varphi) \propto \exp \left( -\frac{\varphi^2}{4\sigma^2} \right)$

Choose $\sigma$ to find minimum energy:

$$E_{\min} = E_J \left( 1 - \left[ 1 - \sqrt{\frac{2E_C}{E_J}} \right]^2 \right) = E_J \left( 1 - \left[ 1 - \frac{\hbar \omega_p}{2E_J} \right]^2 \right)$$

First order in $E_J$:

$$E_{\min} \approx 0 \text{ for } E_C \ll E_J$$

$$\hbar \omega_p = \sqrt{8E_J E_C}$$

\[ \frac{E_C}{E_J} = 0.1 \]

$$E_{\min} = 0.1 E_J$$
3.6.2 Limiting Cases: The Phase and Charge Regime

The phase regime

Tunneling coupling \( \propto \exp \left( -\frac{2E_J - E}{\hbar \omega_p} \right) \) \( \rightarrow \) Very small since \( \hbar \omega_p \ll E_J \)

\( \rightarrow \) Tunneling splitting of low lying states is exponentially small

\[
E_{\text{min}} = E_J \left( 1 - \left[ 1 - \sqrt{\frac{2E_C}{E_J}} \right]^2 \right) = E_J \left( 1 - \left[ 1 - \frac{\hbar \omega_p}{2E_J} \right]^2 \right)
\]

Graph showing the energy levels and tunneling coupling for different values of \( Q/2e \) and \( \varphi/2\pi \).
3.6.2 Limiting Cases: The Phase and Charge Regime

The charge regime

\[ \hbar \omega_p \gg E_{J0}, E_C \gg E_{J0} \rightarrow \text{Charge } Q \text{ (momentum) is good quantum number} \]

Kinetic energy \( \propto E_c \left( \frac{d\varphi}{dt} \right)^2 \) dominates
- Complete delocalization of phase
- Wave function should approach constant value, \( \Psi(\varphi) \approx \text{const.} \)
- Large phase fluctuations, small charge fluctuations \( (\Delta Q \cdot \Delta \varphi \geq 2e) \)

Hamiltonian

\[ \mathcal{H} = -4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi) \]

Appropriate trial function:

\[ \Psi(\varphi) \propto (1 - \alpha \cos \varphi) \quad \alpha \ll 1 \]

Approximate ground state energy

\[ E_{\text{min}} \simeq E_{J0} \left(1 - \frac{E_{J0}}{8E_C} \right) = E_{J0} \left(1 - \frac{E_{J0}^2}{(\hbar \omega_p)^2} \right) \]

second order in \( E_{J0} \)
3.6.2 Limiting Cases: The Phase and Charge Regime

The charge regime

\[ E_{\text{min}} \approx E_{J0} \left( 1 - \frac{E_{J0}}{8E_C} \right) = E_{J0} \left( 1 - \frac{E_{J0}^2}{(\hbar \omega_p)^2} \right) \]

\[ \frac{E_C}{E_{J0}} = 2.5 \]
\[ \Rightarrow E_{\text{min}} = 0.95 E_{J0} \]

→ Periodic potential is weak
→ Strong coupling between neighboring phase states → Broad bands
→ Compare to electrons moving in strong (phase regime) or weak (charge regime) periodic potential of a crystal
### 3.6.3 Coulomb and Flux Blockade

**Coulomb blockade in normal metal tunnel junctions**

Voltage $V \rightarrow$ Charge $Q = CV$, energy $E = \frac{Q^2}{2C}$

**Single electron tunneling**

→ Charge on one electrode changes to $Q - e$
→ Electrostatic energy $E' = \frac{(Q-e)^2}{2C}$
→ Tunneling only allowed for $E' \leq E$
→ **Coulomb blockade**: Need $|Q| \geq e/2$ or $|V| \geq V_{CB} = V_c = e/2C$

**Observation of CB requires small thermal fluctuations**

→ $E_C = \frac{e^2}{2C} > k_B T \Rightarrow C < \frac{e^2}{2k_B T}$
→ $C \simeq 1 \text{ fF at } T = 1 \text{ K, } d = 1 \text{ nm, and } \varepsilon = 5 \Rightarrow A \leq 0.02 \mu\text{m}^2 \rightarrow$ **Small junctions!**

**Observation of CB requires small quantum fluctuations**

→ Quantum fluctuations due to Heisenberg principle $\Rightarrow \Delta E \cdot \Delta t \geq \hbar$
→ Finite tunnel resistance
→ $\tau_{RC} = RC$ (decay of charge fluctuations)
→ $\Delta t = 2\pi RC$, $\Delta E = \frac{e^2}{2C} \Rightarrow R \geq \frac{\hbar}{e^2} = R_K = 24.6 \text{ k}\Omega \rightarrow$ Typically satisfied
3.6.3 Coulomb and Flux Blockade

Coulomb blockade in superconducting tunnel junctions

\[ \frac{Q^2}{2C} > k_B T, eV \quad (Q = 2e) \rightarrow \text{No flow of Cooper pairs} \]

Threshold voltage \( \rightarrow |V| \geq V_{CB} = V_c = \frac{2e}{2C} = \frac{e}{C} \)

Coulomb blockade \( \rightarrow \text{Charge is fixed, phase is completely delocalized} \)
3.6.3 Coulomb and Flux Blockade

Phase or flux blockade in a Josephson junction

Current $I \rightarrow$ Flux $\Phi = LI$, energy $E = \Phi^2/2L$

$\rightarrow$ Phase is blocked due to large $E_{J0} = \Phi_0 I_c/2\pi$

$\rightarrow$ $I_c$ takes the role of $V_{CB}$

$\rightarrow$ Phase change of $2\pi$ equivalent to flux change of $\Phi_0$

$\rightarrow$ Flux blockade $|I| \geq I_{FB} = I_c = \frac{(\Phi_0/2\pi)}{L_c}$

$\rightarrow$ Analogy to CB $\rightarrow I \leftrightarrow V$, $2e \leftrightarrow \frac{\Phi_0}{2\pi}$, $C \leftrightarrow L$

In presence of fluctuations we need

$\rightarrow E_{J0} \gg k_B T$ (large junction area)

$\rightarrow$ And $\Delta E \cdot \Delta t \geq h$ with $\Delta t = 2\pi \frac{L}{R}$ and $\Delta E = 2E_{J0} \rightarrow R \leq \frac{h}{(2e)^2} = \frac{1}{4} R_K$
### 3.6.4 Coherent Charge and Phase States

**Coherent charge states**

Island charge continuously changed by gate

\[ E_{\text{min}} \leq E \leq E_{\text{c}} \]

\[ E_{\text{c}} = \frac{Q^2}{2C} \]

\[ E_{\text{J0}} \]

\[ Q = 2e n \]

**Independent charge states** \((E_{\text{J0}} = 0)\)

\[ \Rightarrow \text{Parabola} \; E(Q) = \left( Q - n \cdot 2e \right)^2 / 2C_{\Sigma} \]

\[ E_{\text{J0}} > 0 \]

\[ \Rightarrow \text{Interaction of } |n\rangle \text{ and } |n + 1\rangle \text{ at the level crossing points } Q = \left( n + \frac{1}{2} \right) \cdot 2e \]

\[ \Rightarrow \text{Avoided level crossing (anti-crossing)} \]

\[ \Rightarrow \text{Coherent superposition states } |\Psi_{\pm}\rangle = \alpha |n\rangle \pm \beta |n + 1\rangle \]
3.6.4 Coherent Charge and Phase States

Coherent charge states

Average charge on the island as a function of the applied gate voltageobilequantized in units of \(2e\) (no coherence yet)

First experimental demonstration of coherent superposition charge states by Nakamura, Pashkin, Tsai (1999, not this picture)
Coherent phase states

→ Interaction of two adjacent phase states
→ Example is rf SQUID

magnetic energy
of flux $\phi = \left( \frac{\Phi_0}{2\pi} \right) \varphi$ in the ring

$$U(\phi) = \frac{(\phi - \phi_{\text{ext}})^2}{2L} + E_{j0} \left( 1 - \cos 2\pi \frac{\phi}{\Phi_0} \right)$$

Tunnel coupling $\rightarrow$ $\Psi_\pm = a|L\rangle \pm b|R\rangle$

Experimental evidence for quantum coherent superposition (Mooij et al., 1999)
3.6.5 Quantum Fluctuations

Violation of conservation of energy on small time scales, obey $\Delta E \cdot \Delta t \geq \hbar$

→ Creation of **virtual excitations**
→ Include Langevin force $I_F$ with adequate statistical properties
→ Fluctuation-dissipation theorem

$$S_I(f) = 2\pi S_I(\omega) = 4 \frac{E(\omega, T)}{R_N}$$

$$E(\omega, T) = \frac{\hbar \omega}{2} + \hbar \omega \frac{1}{\exp \left( \frac{\hbar \omega}{k_B T} \right) - 1} = \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right)$$

- vacuum fluctuations
- occupation probability of oscillator (Planck distribution)

→ Transition from “thermal” Johnson-Nyquist noise to quantum noise:

**classical limit** ($\hbar \omega, eV \ll k_B T$):

$$S_I(\omega) = \frac{1}{2\pi} \frac{4k_B T}{R_N}$$

**quantum limit** ($\hbar \omega, eV \gg k_B T$):

$$S_I(\omega) = \frac{1}{2\pi} \frac{2\hbar \omega}{R_N} = \frac{1}{2\pi} \frac{2eV}{R_N}$$
3.6.6 Macroscopic Quantum Tunneling

Escape of the “phase particle” from minimum of washboard potential by tunneling

→ Macroscopic, i.e., phase difference is tunneling (collective state)
→ States easily distinguishable

Competing process
→ Thermal activation
→ Low temperatures

Neglect damping
Dc-bias → Term $-\frac{hI\phi}{2e}$ in Hamiltonian

Curvature at potential minimum:
$$\frac{\partial^2 U}{\partial \phi^2} = E_{J0} \sqrt{1 - i^2} \quad i = I/I_c$$

(Classical) small oscillation frequency:
$$\omega_A = \omega_0 (1 - i^2)^{1/4} \quad \text{(attempt frequency)}$$
Quantum mechanical treatment

→ Tunnel coupling of bound states to outgoing waves → Continuum of states
→ But only states corresponding to quasi-bound states have high amplitude
→ In-well states of width $\Gamma = \hbar / \tau$ ($\tau = \text{lifetime for escape}$)

Determination of wave functions

→ Wave matching method
→ Exponential prefactor within WKB approximation
→ Decay in barrier:

$$|\psi(x)|^2 \propto \exp \left\{ -\frac{2}{\hbar} \int_{\text{II}} \sqrt{2M[V(x) - E]} \, dx \right\}$$

decay of wave function of particle with mass M and energy E

for $U(\varphi) \gg E_0 = \hbar \omega_A / 2$

$$|\psi(\varphi)|^2 \propto \exp \left\{ -\frac{2}{\hbar} \int_{\text{II}} \sqrt{2} \left( \frac{\hbar}{2e} \right)^2 C \left[ U(\varphi) - \frac{\hbar \omega_A}{2} \right] \, d\varphi \right\}$$

mass effective barrier height
### 3.6.6 Macroscopic Quantum Tunneling

**Constant barrier height**

\[ |\psi(\varphi)|^2 \propto \exp \left\{ -\sqrt{\frac{U_0}{E_C}} \Delta \varphi \right\} \Rightarrow \Gamma = \frac{\omega_A}{2\pi} \exp \left\{ -\sqrt{\frac{U_0}{E_C}} \Delta \varphi \right\} \]

**Escape rate**

**Increasing bias current**

\[ U_0 \approx 2E_0 (1 - i^2)^{3/2} \quad \text{and} \quad \Delta \varphi \approx \pi \sqrt{1 - i^2} \]

**Temperature** \( T^\ast \) where \( \Gamma_{\text{tunnel}} = \Gamma_{TA} \approx \exp \left( -\frac{U_0}{k_B T} \right) \)

**For** \( I \approx 0 \):

\[ U_0 \approx 2E_0 \quad \hbar \omega_p = \sqrt{8E_0 E_C} \approx 2\sqrt{U_0 E_C} \quad \Delta \varphi \approx \pi \]

\[ \Rightarrow \Gamma = \frac{\omega_p}{2\pi} \exp \left\{ -2\pi \frac{U_0}{\hbar \omega_p} \right\} \Rightarrow k_B T^\ast \approx \frac{\hbar \omega_p}{2\pi} \]

**For** \( I > 0 \):

\[ \sqrt{U_0} \propto (1 - i^2)^{3/4} \quad \Delta \varphi \propto (1 - i^2)^{1/2} \]

\[ \Rightarrow k_B T^\ast \approx \frac{\hbar \omega_A}{2\pi} = \frac{\hbar \omega_p}{2\pi} (1 - i^2)^{1/4} \]

**For** \( \omega_p \approx 10^{11} \text{ s}^{-1} \)

\[ T^\ast \approx 100 \text{ mK} \]

**very small for** \( i \approx 0 \)**
3.6.6 Macroscopic Quantum Tunneling

Additional topic: Effect of damping

Junction couples to the environment (e.g., Caldeira-Leggett-type heat bath)

→ Crossover temperature \( k_B T^* \approx \frac{\hbar \omega_R}{2\pi} \) with \( \omega_R = \omega_A \left( \sqrt{1 + \alpha^2} - \alpha \right) \) and \( \alpha \equiv \frac{1}{2R_N C \omega_A} \)

→ Strong damping \( \alpha \gg 1 \) \( \omega_R \ll \omega_A \) → Lower \( T^* \)

→ Damping suppresses MQT


Phase diffusion by MQT

See lecture notes
Classical description only in the phase regime (large junctions): $E_C \ll E_{J0}$

For $E_C \gg E_{J0}$: quantum description (negligible damping):

$$\mathcal{H} = 4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi)$$

Phase difference $\varphi$ and Cooper pair number $N = \frac{Q}{2e}$ are canonically conjugate variables

$$[\varphi, \frac{\hbar}{2e}Q] = i\hbar \quad \Rightarrow \Delta N \cdot \Delta \varphi \geq 1$$

Phase regime: $\Delta \varphi \to 0$ and $\Delta N \to \infty$

Charge regime: $\Delta N \to 0$ and $\Delta \varphi \to \infty$

Charge regime at $T = 0$

→ Coulomb blockade → Tunneling only for $V_{CB} \geq \frac{e}{C}$

Flux regime at $T = 0$

→ Flux blockade → Flux motion only for $I_{FB} \geq \frac{\Phi_0}{2\pi L_c}$

At $I < I_c$

→ Escape out of the washboard by thermal activation or macroscopic quantum tunneling

TA-MQT crossover temperature $T^*$

$$k_B T^* \simeq \frac{\hbar \omega_A}{2\pi} = \frac{\hbar \omega_p}{2\pi} \left[1 - \left(\frac{l}{l_c}\right)^2\right]^{1/4}$$