Chapter 3

Physics of Josephson Junctions:

The Voltage State
3. Physics of Josephson Junctions: The Voltage State

- for $I > I_s m$:
  - finite junction voltage
    $\Rightarrow$ phase difference $\varphi$ evolves in time: $d\varphi/dt \propto V$
  - finite voltage state of junction corresponds to dynamic state

- only part of the total current is carried by the Josephson current
  $\Rightarrow$ additional resistive channel, capacitive channel, and noise

- questions:
  - how does the phase dynamics look like?
  - current-voltage characteristics for $I > I_s m$?
  - what is the influence of the resistive damping?
3.1 The Basic Equation of the Lumped Josephson Junction

3.1.1 The Normal Current: Junction Resistance

- for \( T > 0 \): finite density of “normal” electrons \( \rightarrow \) \textbf{quasiparticles}
  - zero-voltage state: no quasiparticle current
  - for \( V > 0 \): quasiparticle current \( \equiv \) normal current \( I_N \) \( \rightarrow \) resistive state

- high temperature close to \( T_c \):
  - for \( T \leq T_c \): \( 2 \Delta(T) \ll k_B T \): (almost) all Cooper pairs are broken up, \textbf{Ohmic IVC}:
    \[
    I_N = G_N V
    \]
    \[
    G_N = 1/R_N: \text{normal conductance}
    \]

- large voltage \( V > V_g = (\Delta_1 + \Delta_2)/e \):
  \( \rightarrow \) external circuit provides energy to break up Cooper pairs \( \rightarrow \) \textbf{Ohmic IVC}

- for \( T \ll T_c \) and \( |V| < V_g \): vanishing quasiparticle density \( \rightarrow \) \textbf{no} normal current
3.1.1 The Normal Current: Junction Resistance

- for $T \ll T_c$ and $|V| < V_g$: IVC depends on **sweep direction** (and external source)
  hysteretic behavior
  current source: $I = I_s + I_N = \text{const.}$

voltage state: $I_s(t) = I_c \sin \phi(t)$ is time dependent
  $\rightarrow I_N$ is time dependent
  $\rightarrow$ junction voltage $V = I_N/G_N$ is time dependent
  $\rightarrow$ IVC: **time averaged voltage**

- equivalent conductance $G_N$ at $T = 0$:
  
  $$G_N(V) = \begin{cases} 
  0 & \text{for } |V| < 2\Delta/e \\
  \frac{1}{R_N} & \text{for } |V| \geq 2\Delta/e 
  \end{cases}$$
3.1.1 The Normal Current: Junction Resistance

• finite temperature: sub-gap resistance $R_{sg}(T)$ for $|V| < V_g$

$$R_{sg}(T) \text{ determined by amount of thermally excited quasiparticles:}$$

$$G_{sg}(T) = \frac{1}{R_{sg}(T)} = \frac{n(T)}{n_{tot}} G_N$$

$n(T)$: density of excited quasiparticles

for $T > 0$ we get:

$$G_N(V, T) = \begin{cases} 
\frac{1}{R_{sg}(T)} & \text{for } |V| < 2\Delta(T)/e \\
\frac{1}{R_N} & \text{for } |V| \geq 2\Delta(T)/e 
\end{cases}$$

nonlinear conductance $G_N(V,T)$

• characteristic voltage ($I_cR_N$ - product):

$$V_c \equiv I_c R_N = \frac{I_c}{G_N}$$

note:
- there may be frequency dependence of normal channel
- normal channel depends on junction type
3.1.2 The Displacement Current: Junction Capacitance

- if \( \frac{dV}{dt} \neq 0 \) \( \Rightarrow \) finite displacement current
  \[ I_D = C \frac{dV}{dt} \]
  
  \( C \): junction capacitance, for planar tunnel junction:
  \[ C = \frac{\varepsilon_0 A_i}{d} \]

- additional current channel

  \[ V = L_c \frac{dI_s}{dt}, \quad I_N = VG_{N}V, \quad I_D = C \frac{dV}{dt} \quad \text{and} \quad L_s = \frac{L_c}{\cos \varphi} \geq L_c, \quad G_N(V,T) = \frac{1}{R_N} \]

\[ I_s \leq \frac{V}{\omega L_c}, \quad I_N \leq \frac{V}{R_N}, \quad I_D \approx \omega C V \]

\[ L_c = \frac{\hbar}{2eI_c} \quad \text{Josephson inductance} \]
3.1.3 Characteristic Times and Frequencies

equivalent circuit: \( L_c, R_N, C \) \( \rightarrow \) 3 characteristic frequencies

- **plasma frequency:** 
  \[ \omega_p = \frac{1}{\tau_p} \equiv \frac{1}{\sqrt{L_c C}} = \sqrt{\frac{2eI_c}{\hbar C}} \]

  scales \( \propto (J_c/C_A)^{0.5} \), \( C_A = C/A \): *specific junction capacitance*

  for \( \omega < \omega_p \): \( I_D < I_s \)

- **\( L_c/R_N \) time constant:** 
  \[ \omega_c = \frac{1}{\tau_c} \equiv \frac{R_N}{L_c} = \frac{2e}{\hbar} V_c = \frac{2\pi}{\Phi_0} V_c \]

  inverse relaxation time in system of normal and supercurrent

  \( \omega_c \) follows from \( V_c \) (2\textsuperscript{nd} Josephson eq.) \( \rightarrow \) *characteristic frequency*

  \( I_N < I_c \) for \( V < V_c \) or \( \omega < \omega_c = R_N/L_c \)

- **\( R_N C \) time constant:** 
  \[ \omega_{RC} = \frac{1}{\tau_{RC}} \equiv \frac{1}{R_N C} = \frac{\omega_p^2}{\omega_c} \]

  \( I_D < I_N \) for \( \omega < 1/\tau_{RC} \)

- **Stewart-McCumber parameter:** (related to quality factor of LCR circuit)

  \[ \beta_C \equiv \frac{\omega_c^2}{\omega_p^2} = \frac{\omega_c}{\omega_{RC}} = \omega_c \tau_{RC} = \frac{2e}{\hbar} I_c R_N^2 C \]
3.1.3 Characteristic Times and Frequencies

**Quality Factor:**

\[ Q = \frac{RC}{\sqrt{LC}} = \frac{\omega_p}{\omega_{RC}} = \frac{\omega_c}{\omega_p} = \sqrt{\beta_c} \]

→ parallel LRC circuit

→ \( Q \) compares *decay of amplitude of oscillations to oscillation period*

→ \( \beta_c \ll 1 \): small capacitance and/or small resistance

  → small \( R_N C \) time constants \((\tau_{RC}\omega_p \ll 1)\)

    → *highly damped* or *overdamped* junctions

→ \( \beta_c \gg 1 \): large capacitance and/or large resistance

  → large \( R_N C \) time constants \((\tau_{RC}\omega_p \gg 1)\)

    → *weakly damped* or *underdamped* junctions
3.1.4 The Fluctuation Current

fluctuation/noise

→ Langevin method: include random source → fluctuating noise current

→ type of fluctuations: white noise, shot noise, 1/f noise

Thermal Noise:

Johnson-Nyquist formula for thermal noise ($k_B T \gg eV, \hbar \omega$):

\[ S_I(f) = \frac{4k_B T}{R_N} \]  
\[ S_V(f) = 4k_B T R_N \]

(current noise power spectral density)

(voltage noise power spectral density)

relative noise intensity (thermal energy/Josephson coupling energy):

\[ \gamma \equiv \frac{k_B T}{E_J} = \frac{2e}{\hbar} \frac{k_B T}{I_c} \]

\[ \Rightarrow \gamma \equiv \frac{I_T}{I_c} \text{ with } I_T = \frac{2e}{\hbar} k_B T \]

$I_T$: thermal noise current

for $T = 4.2$ K: $I_T \approx 0.15 \ \mu A$
3.1.4 The Fluctuation Current

**Shot Noise:**

Schottky formula for shot noise \((eV \gg k_B T, V > 0.5 \text{ mV} @ 4.2 \text{ K}):\)

\[ S_I(f) = 2eI_N \]

→ random fluctuations, **discreteness of charge carriers**
→ Poisson process → Poissonian distribution
→ strength of fluctuations: variance: \(\Delta I^2 \equiv \langle (I - \langle I \rangle)^2 \rangle\)
→ variance depends on frequency → use noise power:

\[ S(f) = \int_{-\infty}^{+\infty} \left( \langle I(t)I(0) \rangle - \langle I(0) \rangle^2 \right) dt \]

includes equilibrium fluctuations (white noise)

**1/f noise:**

→ dominant at **low frequencies**
→ physical nature often **unclear**
→ Josephson junctions: dominant below about 1 Hz - 1 kHz → not considered here
3.1.5 The Basic Junction Equation

Kirchhoff’s law: \[ I = I_s + I_N + I_D + I_F \]

\&

voltage-phase relation: \[ \frac{d\varphi}{dt} = 2eV/\hbar \]

→ basic equation of Josephson junction

\[ I = I_c \sin \varphi + G_N(V)V + C \frac{dV}{dt} + I_F \]

\[ I = I_c \sin \varphi + G_N(V) \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} + C \frac{\Phi_0}{2\pi} \frac{d^2\varphi}{dt^2} + I_F \]

nonlinear differential equation with nonlinear coefficients

→ complex behavior, numerical solution

→ use approximations (simple models)
3.2 The Resistively and Capacitively Shunted Junction Model

- **Resistively and Capacitively Shunted Junction model (RCSJ):**
  
an approximation: \( G_N(V) = G = 1/R = \text{const.} \)

Josephson junction:

\[
L_s = L_c / \cos \varphi \quad \text{with} \quad L_c = \hbar / 2e I_c
\]

\( R = 1/G \): junction normal resistance

**nonlinear differential equation:**

\[
\left( \frac{\hbar}{2e} \right) C \frac{d^2 \varphi}{dt^2} + \left( \frac{\hbar}{2e} \right) \frac{1}{R} \frac{d \varphi}{dt} + I_c \left[ \sin \varphi - \frac{l}{l_c} + \frac{l_F(t)}{l_c} \right] = 0
\]

\[
\Rightarrow \left( \frac{\hbar}{2e} \right)^2 C \frac{d^2 \varphi}{dt^2} + \left( \frac{\hbar}{2e} \right)^2 \frac{1}{R} \frac{d \varphi}{dt} + \frac{d}{d\varphi} \left\{ E_{J0} \left[ 1 - \cos \varphi - i \varphi + i_F(t) \varphi \right] \right\} = 0
\]

- **compare motion** of gauge invariant phase difference to that of **particle with mass** \( M \) and **damping** \( \eta \) in **potential** \( U \):

\[
M \frac{d^2 x}{dt^2} + \eta \frac{dx}{dt} + \nabla U = 0
\]

with \( M = \left( \frac{\hbar}{2e} \right)^2 C \quad \eta = \left( \frac{\hbar}{2e} \right)^2 \frac{1}{R} \)

\[
U = E_{J0} \left[ 1 - \cos \varphi - i \varphi + i_F(t) \varphi \right]
\]

**tilted washboard potential**
3.2 The RCSJ Model

finite tunneling probability: \( \rightarrow \) macroscopic quantum tunneling (MQT)

escape by thermal activation \( \rightarrow \) thermally activated phase slips

normalized time: \( \tau \equiv \frac{t}{\tau_c} = \frac{t}{2eI_c R/\hbar} \)

Stewart-McCumber parameter: \( \beta_c \equiv \frac{\omega_c^2}{\omega_p^2} = \frac{2e}{\hbar} I_c R^2 N_C \)

motion of \( \varphi \) in tilted washboard potential

plasma frequency:

neglect damping, zero driving and small amplitudes (\( \sin \varphi \simeq \varphi \)): \( \beta_c \frac{d^2 \varphi}{d\tau^2} + \varphi = 0 \)

solution: \( \varphi = c \cdot \exp \left( i \frac{\tau}{\sqrt{\beta_c}} \right) = c \cdot \exp \left( i \frac{t}{\sqrt{\beta_c \tau_c}} \right) = c \cdot \exp (i \omega_p t) \)

plasma frequency \( \rightarrow \) oscillation frequency around potential minimum
3.2 The RCSJ Model

The pendulum analogue:
- **physical pendulum:**
  - mass $m$, length $\ell$, deflection angle $\theta$,
  - torque $D$ parallel to rotation axis
- **restoring torque:** $\ell m g \sin \Theta$

**equation of motion:**
$$D = \Theta \ddot{\theta} + \Gamma \dot{\theta} + m g \ell \sin \theta$$

- $\Theta = m \ell^2$: moment of inertia
- $\Gamma$: damping constant

analogy to Josephson junction:
- $I \leftrightarrow D$
- $I_c \leftrightarrow m g \ell$
- $\Phi_0/2\pi R \leftrightarrow \Gamma$
- $C \Phi_0/2\pi \leftrightarrow \Theta$

- gauge invariant phase difference $\varphi \leftrightarrow$ angle $\theta$

for $D = 0$: oscillations around equilibrium with
- $\omega = (g/\ell)^{1/2} \leftrightarrow$ plasma frequency $\omega_p = (2\pi I_c/\Phi_0 C)^{1/2}$
- **finite** torque $\rightarrow$ finite $\theta_0 \rightarrow$ finite $\varphi_0$
- **large** torque (deflection $> 90^\circ$) $\rightarrow$ rotation of the pendulum $\rightarrow$ finite **voltage** state
3.2.1 Underdamped and Overdamped Josephson Junctions

underdamped junction:

\[ \beta_c = 2eI_c R^2 C / \hbar \ll 1 \]

\[ \rightarrow \text{capacitance & resistance small} \]
\[ \rightarrow M \text{ small, } \eta \text{ large} \]
\[ \rightarrow \text{non-hysteretic IVC} \]

(once the phase is moving, the potential has to be tilt back almost into the horizontal position to stop its motion)

overdamped junction:

\[ \beta_c = 2eI_c R^2 C / \hbar \gg 1 \]

\[ \rightarrow \text{capacitance & resistance large} \]
\[ \rightarrow M \text{ large, } \eta \text{ small} \]
\[ \rightarrow \text{hysteretic IVC} \]
3.3 Response to Driving Sources

3.3.1 Response to a dc Current Source

- **time averaged voltage:**

\[
\langle V \rangle = \frac{1}{T} \int_0^T V(t) \, dt = \frac{1}{T} \int_0^T \frac{\hbar}{2e} \frac{d\varphi}{dt} \, dt = \frac{1}{T} \frac{\hbar}{2e} [\varphi(T) - \varphi(0)] = \frac{\Phi_0}{T}
\]

- **total current** must be **constant** (neglecting the fluctuation source):

\[
l = l_s(t) + l_N(t) + l_D(t) = I_c \sin \varphi(t) + \frac{V(t)}{R} + C \frac{dV(t)}{dt} = \text{const}
\]

where: \( \varphi(t) = \int_0^t \frac{2e}{\hbar} V(t) \, dt \)

- for \( I > I_c \): part of the current must flow as \( l_N \) or \( l_D \)
  - finite junction voltage \( |V| > 0 \)
  - \( |V| > 0 \) \( \rightarrow \) time varying \( l_s \)
  - \( l_N + l_D \) is varying in time
  - **time varying voltage**, complicated non-sinusoidal oscillations of \( l_s \)
  - oscillating voltage has to be calculated self-consistently
  - oscillation frequency: \( f = \langle V \rangle/\Phi_0 \)
3.3.1 Response to a dc Current Source

- **for** $I \geq I_c$:  
  highly **non-sinusoidal** oscillations
- **long** oscillation period
- $\langle V \rangle \propto 1/T$ : small

- **for** $I \gg I_c$:
  - almost all current flows as normal current
  - junction voltage is about constant
  - oscillations of Josephson current are almost sinusoidal
  - time averaged Josephson current almost zero
  - linear/Ohmic IVC

→ analogy to pendulum
3.3.1 Response to a dc Current Source

Current-Voltage Characteristics:

**strong damping:** $\beta_c \ll 1$, neglecting noise current:

\[
\beta_c \frac{d^2 \varphi}{d\tau^2} + \frac{d\varphi}{d\tau} + \sin \varphi - i = 0
\]

- **for** $i < 1$: only supercurrent, $\varphi = \sin^{-1} i$ is a solution, junction voltage = zero
- **for** $i > 1$: finite voltage, temporal evolution of the phase

\[
d\tau = \frac{d\varphi}{i - \sin \varphi}
\]

integration using:

\[
\int \frac{dx}{a - \sin x} = \frac{2}{\sqrt{a^2 - 1}} \tan^{-1} \left( \frac{-1 + a \tan(x/2)}{\sqrt{a^2 - 1}} \right)
\]

gives

\[
\tau - \tau_0 = \frac{2}{\sqrt{i^2 - 1}} \tan^{-1} \left( \frac{-1 + i \tan(\varphi/2)}{\sqrt{i^2 - 1}} \right)
\]

\[
\Rightarrow \varphi(t) = 2 \tan^{-1} \left\{ \sqrt{1 - \frac{1}{i^2}} \tan \left( \frac{t \sqrt{i^2 - 1}}{2 \tau_c} \right) + \frac{1}{i} \right\}
\]

periodic function with period:

\[
T = \frac{2\pi \tau_c}{\sqrt{i^2 - 1}}
\]

setting $\tau_0 = 0$ and using $\tau = t/\tau_c$
3.3.1 Response to a dc Current Source

with \( \langle V(t) \rangle = \frac{1}{T} \int_{0}^{T} V(t) \, dt = \frac{\Phi_0}{T} \)

and \( \tau_c = \frac{\Phi_0}{2\pi} \frac{1}{I_c R} \)

we get for \( i > 1 \):

\[ \langle V(t) \rangle = I_c R \sqrt{\left(\frac{I}{I_c}\right)^2 - 1} \]
3.3.1 Response to a dc Current Source

Current-Voltage Characteristics:

Weak damping: $\beta_c \gg 1$, neglecting noise current:

$\omega_{RC} = 1/R_NC$ is very small

$\rightarrow$ large $C$ is effectively shunting oscillating part of junction voltage $\rightarrow V(t) \approx \overline{V}$

$\rightarrow$ time evolution of the phase:

$$\varphi(t) = \frac{2e}{\hbar} \overline{V} \ t + \text{const}$$

$\rightarrow$ almost sinusoidal oscillation of Josephson current:

$$\overline{l_s(t)} = l_c \sin \left( \frac{2e}{\hbar} \overline{V} t + \text{const} \right) \approx 0$$

down to $\overline{V} \simeq \hbar \omega_{RC}/e \ll V_c = l_c R_N$

$\Rightarrow \overline{l} = I_N(\overline{V}) = \frac{\overline{V}}{R}$

corresponding current $\ll l_c \rightarrow$ hysteretic IVC
3.3.1 Response to a dc Current Source

**additional topic:** intermediate damping: $\beta_c \sim 1$

- numerically solve IVC
- increasing McCumber parameter $\rightarrow$ increasing hysteresis
- decreasing return-current $I_R$

- $I_R$ given by tilt of washboard where:
  - energy dissipated in advancing to next minimum $=$ work done by drive current

- calculation of $I_R$ for $\beta_c \gg 1$:
  - for $I \geq I_R$: normal current can be neglected junction energy:

$$E = \frac{\phi_0}{2\pi} \int_0^\varphi I d\varphi' = \frac{\phi_0}{2\pi} \int_0^\varphi \left( I_c \sin \varphi' + C \frac{\phi_0}{2\pi} \frac{d^2 \varphi'}{dt^2} \right) d\varphi'$$

$$= E_{J0} \left\{ \frac{1}{2} \left( \frac{d\varphi}{dt} \right)^2 + (1 - \cos \varphi) \right\}$$

energy dissipation within RCSJ model:

$$W_{\text{diss}} = \int_0^T I_N V dt = \int_0^T \frac{\hbar}{2e} \frac{d\varphi}{dt} dt = \int_0^{2\pi} \frac{V \hbar}{2e} d\varphi = \int_0^{2\pi} \left( \frac{d\varphi}{dt} \right) \left( \frac{\hbar}{2e} \right)^2 \frac{1}{R} d\varphi$$

$$= \frac{\phi_0 V_p}{2\pi R} \int_0^{2\pi} \left\{ 2 \left( \frac{E}{E_{J0}} - 1 + \cos \varphi \right) \right\}^{1/2} d\varphi$$

$$V_p = \omega_p \frac{\phi_0}{2\pi} = \omega_p \frac{\hbar}{2e} = \frac{V_c}{\sqrt{\beta_c}}$$
### 3.3.1 Response to a dc Current Source

**Resistive state:** minimum junction kinetic energy must be positive \( E \geq 2E_{J0} \)

Limit \( I = I_R \rightarrow E = 2E_{J0} \)

Energy dissipation:

\[
W_{diss} = 4 \frac{\Phi_0}{\pi} \frac{I_c}{\sqrt{\beta_c}}
\]

Work done by the current source:

\[
\int F dx = \int_0^{2\pi} I_R V dt = \frac{\Phi_0}{2\pi} I_R \int_0^{2\pi} \dot{\varphi} dt = I_R \Phi_0
\]

Then:

\[
\frac{I_R}{I_c} = \frac{4}{\pi} \frac{1}{\sqrt{\beta_c}}
\]

Valid for \( \beta_c \gg 1 \)
Summary

plasma frequency:
\[ \omega_p = \sqrt{\frac{2eI_c}{\hbar C}} \]

\( L_c/R_N \) time constant:
\[ \omega_c = \frac{R_N}{L_c} = \frac{2e}{\hbar} V_c \]

\( R_N C \) time constant:
\[ \omega_{RC} = \frac{1}{R_N C} = \frac{\omega_p^2}{\omega_c} \]

Stewart-McCumber parameter:
\[ \beta_C = Q^2 \equiv \frac{\omega_c^2}{\omega_p^2} = \frac{\omega_c}{\omega_{RC}} = \omega_c \tau_{RC} = \frac{2e}{\hbar} I_c R_N^2 C \]

\[ \Rightarrow \left( \frac{\hbar}{2e} \right)^2 C \frac{d^2 \varphi}{dt^2} + \left( \frac{\hbar}{2e} \right)^2 \frac{1}{R} \frac{d \varphi}{dt} + \frac{d}{d \varphi} \left\{ E_{J0} \left[ 1 - \cos \varphi - i \varphi + i_F(t) \varphi \right] \right\} = 0 \]

\[ \tau \equiv \frac{t}{\tau_c} \]

\[ \beta_C \frac{d^2 \varphi}{d \tau^2} + \frac{d \varphi}{d \tau} + \sin \varphi - i - i_F(\tau) = 0 \]
Summary

time averaged voltage:  \( \langle V \rangle = \frac{1}{T} \int_0^T V(t) \, dt = \frac{\Phi_0}{T} \)

strong damping:  \( \beta_c \ll 1 \), neglecting noise current:

\[
\beta_c \frac{d^2 \varphi}{d\tau^2} + \frac{d \varphi}{d\tau} + \sin \varphi - i = 0
\]

\[
\frac{l_R}{l_c} = \frac{4}{\pi} \frac{1}{\sqrt{\beta_c}}
\]
3.3.2 Response to a dc Voltage Source

**phase** evolves **linearly** in time:

\[ \varphi(t) = \frac{2e}{\hbar} V_{dc} t + \text{const} \]

→ Josephson current \( I_s \) **oscillates sinusoidally**
→ time average of \( I_s \) is zero
→ \( I_D = 0 \) since \( dV/dt = 0 \)
→ **total** current carried by **normal current** \( \Rightarrow \) IVC:

\[ I = \frac{V_{dc}}{R_N} \]

- RCSJ model: **ohmic** IVC
- in more general \( R = R_N(V) \): **nonlinear** IVC
3.3.2 Response to ac Driving Sources

- response to an ac voltage source
- weak damping, $\beta_c \gg 1$

\[ V(t) = V_{dc} + V_1 \cos \omega_1 t \]

integrating the voltage-phase relation:

\[ \varphi(t) = \varphi_0 + \frac{2\pi}{\Phi_0} V_{dc} t + \frac{2\pi}{\Phi_0} \frac{V_1}{\omega_1} \sin \omega_1 t \]

current-phase relation:

\[ I_s(t) = I_c \sin \left\{ \varphi_0 + \frac{2\pi}{\Phi_0} V_{dc} t + \frac{2\pi}{\Phi_0} \frac{V_1}{\omega_1} \sin \omega_1 t \right\} \]

superposition of linearly increasing $\omega_{dc} t = \left(2\pi V_{dc}/\Phi_0\right)t$ and sinusoidally varying phase

→ current $I_s(t)$ and ac voltage $V_1$ have different frequencies
→ origin: nonlinear current-phase relation
### 3.3.2 Response to ac Driving Sources

Some mathematics for the analysis of the time-dependent Josephson current:

**Fourier-Bessel series identity:**

\[ e^{ib\sin x} = \sum_{n=-\infty}^{+\infty} J_n(b)e^{inx} \]

And:

\[ \sin(a + b \sin x) = \Im \left\{ e^{i(a+b\sin x)} \right\} \]

\[ \Rightarrow e^{i(a+b\sin x)} = \sum_{n=-\infty}^{+\infty} J_n(b)e^{i(a+nx)} = \sum_{n=-\infty}^{+\infty} (-1)^n J_n(b)e^{i(a-nx)} \]

\[ \Rightarrow \sin(a + b \sin x) = \sum_{n=-\infty}^{+\infty} (-1)^n J_n(b) \sin(a - nx) \]

With \( x = \omega_1 t, \ b = 2\pi V_1/\Phi_0 \omega_1 \) and \( a = \varphi_0 + \omega_{dc} t = \varphi_0 + \frac{2\pi}{\Phi_0} V_{dc} t \)

\[ I_s(t) = I_c \sum_{n=-\infty}^{+\infty} (-1)^n J_n \left( \frac{2\pi V_1}{\Phi_0 \omega_1} \right) \sin [(\omega_{dc} - n\omega_1)t + \varphi_0] \]

→ Frequency \( \omega_{dc} \) couples to *multiples* of the driving frequency.
3.3.2 Response to ac Driving Sources

\[ I_s(t) = I_c \sum_{n=-\infty}^{+\infty} (-1)^n J_n \left( \frac{2\pi V_1}{\Phi_0 \omega_1} \right) \sin \left[ (\omega_{dc} - n\omega_1) t + \varphi_0 \right] \]

Shapiro steps: \textit{ac voltage results in dc supercurrent} if \([ \ldots ]\) is time independent

\[ \omega_{dc} = n\omega_1 \text{ or } V_{dc} = V_n = n \frac{\Phi_0}{2\pi} \omega_1 \]

amplitude of average dc current for a specific \(n\) (step number):

\[ |\langle I_s \rangle_n| = I_c \left| J_n \left( \frac{2\pi V_1}{\Phi_0 \omega_1} \right) \right| \]

for \(V_{dc} \neq V_n\): \([\ldots]\) is time dependent

\[ \Rightarrow \text{sum of sinusoidally varying terms} \]
\[ \Rightarrow \text{time average is zero} \Rightarrow \text{vanishing dc component} \]

\[ \langle I \rangle = \frac{V_{dc}}{R_N} + \left( \frac{V_1}{R_N} \right) \cos \omega_1 t = \frac{V_{dc}}{R_N} \]
3.3.2 Response to ac Driving Sources

→ Ohmic dependence with sharp **current spikes** at $V_{dc} = V_n$
→ current spike amplitude depends on ac voltage amplitude
→ $n^{th}$ step: *phase locking of the junction to the $n^{th}$ harmonic*

\[
V_n = \frac{n\Phi_0}{2\pi\omega_1}
\]

\[
|\langle I_s \rangle_n| = I_c \left| J_n \left( \frac{2\pi V_1}{\Phi_0 \omega_1} \right) \right|
\]

example: $\omega_1/2\pi = 10$ GHz
constant dc current at $V_{dc} = 0$ and $V_n = n\omega_1 \times \Phi_0/2\pi \approx n \times 20 \mu V$
3.3.2 Response to ac Driving Sources

- **response to an ac current source**, **strong damping**, $\beta_c \ll 1$ (experimentally relevant)

Kirchhoff’s law (neglecting $I_D$):

$$I_c \sin \varphi + \frac{1}{R_N} \frac{\phi_0}{2\pi} \frac{d\varphi}{dt} = I_{dc} + I_1 \sin \omega_1 t$$

Difficult to solve $\to$ qualitative discussion with **washboard potential**:

- increase $I_{dc}$ at constant $I_1$:
  - zero-voltage state for $I_{dc} + I_1 \leq I_c$
  - voltage state for $I_{dc} + I_1 > I_c$ $\Rightarrow$ **complicated dynamics**

for $V_n = n \cdot \omega_1 \cdot \Phi_0 / 2\pi$:

- motion of phase particle **synchronized** by ac driving

Assumption: during each ac-cycle phase the particle moves down $n$ minima

$\to$ resulting **phase change**:

$$\dot{\varphi} = n \frac{2\pi}{T} = n\omega_1$$

$\Rightarrow$ **average dc voltage**:

$$\langle V \rangle = n \frac{\Phi_0}{2\pi} \omega_1 = V_n$$

**synchronization** of phase dynamics with external ac source (for certain bias current interval)
3.3.2 Response to ac Driving Sources

Experimental IVCs obtained for an underdamped and overdamped Niobium Josephson junction under microwave radiation.
3.3.4 Photon-Assisted Tunneling

- superconducting tunnel junction: highly nonlinear $R(V)$, sharp step at $V_g = 2\Delta/e$
  - use $I_{qp}(V)$, include effect of ac source on qp-tunneling

- Model of Tien and Gordon:
  - ac driving shifts levels in electrode up and down
    - qp-energy: $E_{qp} + eV_1\cos \omega_1 t$
    -qm phase factor:
      $$\exp \left( -\frac{i}{\hbar} \int (E_{qp} + eV_1 \cos \omega_1) dt \right) = \exp \left( -\frac{i}{\hbar} E_{qp} t \right) \cdot \exp \left( -i\frac{eV_1}{\hbar \omega_1} \sin \omega_1 t \right)$$

- Bessel function identity for $V_1$-term: sum of terms $\mathcal{J}_n(eV_1/\hbar \omega_1) e^{-in\omega_1 t}$
  - splitting of qp-levels in many levels $E_{qp} \pm n\hbar \omega_1$

  tunneling current:
  $$I_{qp}(V) = \sum_{n=-\infty}^{+\infty} \mathcal{J}_n^2 \left( \frac{eV_1}{\hbar \omega_1} \right) I_{qp}^0 (V + n\hbar \omega_1 / e)$$

  - sharp increase of the qp tunneling current at the gap voltage is broken up into many steps of smaller current amplitude at voltages $V_g \pm n\hbar \omega_1 / e$
3.3.4 Photon-Assisted Tunneling

qp IVC of a niobium SIS Josephson junction **without and with microwave irradiation**

frequency $2\pi \omega_1 = 230$ GHz corresponding to $\hbar \omega_1/e \approx 950$ $\mu$V

**Shapiro steps:**

$V_n = n \frac{\hbar}{2e} \omega_1$

**qp steps:**

$V_n = n \frac{\hbar}{e} \omega_1$

(note that qp steps have no constant voltage and different amplitude)
3.4 Additional Topic: Effect of Thermal Fluctuations

- **thermal** fluctuations with correlation function:

- **small** fluctuations
  - phase fluctuations around equilibrium

- **large** fluctuations
  - increase probability for escape out of potential well

  escape → **rates** $\Gamma_{n\pm 1}$
  - escape to next minimum
  - phase change of $2\pi$
  - for $I > 0$: $\Gamma_{n+1} > \Gamma_{n-1}$ → $\langle d\varphi/dt \rangle > 0$

- **Langevin equation** for RCSJ model:

  \[
  l = l_c \sin \varphi + \frac{1}{R_N} \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} + C \frac{\Phi_0}{2\pi} \frac{d^2\varphi}{dt^2} + I_F
  \]

  equivalent to **Fokker-Planck** equation:

  \[
  \frac{1}{\omega_c} \frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial \varphi} (\sigma v) + \frac{1}{\beta_C} \frac{\partial}{\partial v} \left( \sigma [f(\varphi) - v] \right) = \frac{\gamma}{\beta_C^2} \frac{\partial^2 \sigma}{\partial v^2}
  \]

  normalized force: $f(\varphi) = -\frac{1}{E_{J0}} \frac{\partial U(\varphi)}{\partial \varphi} = \frac{l}{l_c} - \sin \varphi$
3.4 Additional Topic: Effect of Thermal Fluctuations

normalized momentum: \( v = \frac{d\varphi/dt}{\omega_c} = \frac{V}{I_c R_N} \)

\( \sigma(v, \varphi, t) \): probability density of finding system at \((v, \varphi)\) at \(t\)

\[
\langle X \rangle(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma(\varphi, v, t) X(\varphi, v, t) d\varphi dv
\]

**small fluctuations** \(\rightarrow\) **static solution:** \( \sigma(v, t) = \mathcal{F}^{-1} \exp \left( -\frac{G(\varphi, \sigma)}{k_B T} \right) \)

with: \( \mathcal{F} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left( -\frac{G(\varphi, \sigma)}{k_B T} \right) d\varphi dv \)

**Boltzmann distribution** \((G = E - Fx: \text{total energy}, E: \text{free energy})\)

constant probability to find system in \(n^{th}\) metastable state: \( p = \int_{-\infty}^{+\infty} d\nu \int_{\varphi \approx \varphi_n}^{+\infty} \sigma(\varphi, \nu) d\varphi \)

**large fluctuations:** \( p \) can change in time: \( \frac{dp}{dt} = (\Gamma_{n+1} - \Gamma_{n-1}) p \)

for \( \Gamma_{n+1} \gg \Gamma_{n-1} \) and \( \omega_A/\Gamma_{n+1} \gg 1 \):

\[
\Gamma_{n+1} = \frac{\omega_A}{2\pi} \exp \left( -\frac{U_0}{k_B T} \right)
\]

\( \omega_A \): attempt frequency

statistical average of variable \(X\)
3.4 Additional Topic: Effect of Thermal Fluctuations

attempt frequency $\omega_A$:

$$\omega_A = \omega_0 = \omega_p (1 - i^2)^{1/4} \quad \text{for} \quad \omega_c \tau \gg 1,$$

$$\omega_A = \tau^{-1} = \omega_c (1 - i^2)^{1/2} \quad \text{for} \quad \omega_c \tau \ll 1$$

for $I = 0$: $\omega_A \rightarrow$ plasma frequency $\omega_p$ (oscillation frequency in potential well)
for $I < I_c$: $\omega_A \sim \omega_p$

**strong damping** ($\beta_c = \omega_c \tau_{RC} \ll 1$): $\omega_p \rightarrow \omega_c$ (frequency of overdamped oscillator)
3.4.1 Underdamped Junctions: $I_c$ Reduction by Premature Switching

for $E_{j0} >> k_B T$: small escape probability $\propto \exp(-U_0(I)/k_B T)$ at each attempt

barrier height: $U_0(I) \approx 2E_{j0} \left(1 - \frac{I}{I_c}\right)^{3/2}$ \quad $\rightarrow 2E_{j0}$ for $I = 0$,
barrier height $\rightarrow 0$ for $I \rightarrow I_c$

escape probability $\rightarrow \omega_A / 2\pi$ for $I \rightarrow I_c$

after escape: junction switches to $IR_N$

experiment:
one measures distribution of escape current $I_M$

$\rightarrow$ width $\delta I$ and mean reduction $\langle \Delta I_c \rangle = I_c - \langle I_M \rangle$

use approximation for $U_0(I)$ and escape rate

$\propto \omega_A / 2\pi \exp(-U_0(I)/k_B T)$:

$$\langle \Delta I_c \rangle = I_c - \langle I_M \rangle \approx I_c \left[ \frac{k_B T}{2E_{j0}} \ln \left( \frac{\omega_p \Delta t}{2\pi} \right) \right]^{2/3}$$

$\rightarrow$ considerable reduction of $I_c$ when $k_B T > 0.05 E_{j0}$
3.4.2 Overdamped Junctions: The Ambegaokar-Halperin Theory

calculate voltage $\langle V \rangle$ induced by thermally activated phase slips as a function of current

important parameter:

$$\gamma_0(T) = \frac{2E_{J_0}(T)}{k_B T} = \frac{\Phi_0 I_c(T)}{\pi k_B T}$$
3.4.2 Overdamped Junctions: The Ambegaokar-Halperin Theory

*Ambegaokar, Halperin*: finite amount of phase slippage
- nonvanishing voltage for \( I \to 0 \)
- phase slip resistance for strong damping \((\beta_c \ll 1)\), for \( U_0 = 2E_{J0} \):

\[
R_p = \lim_{I \to 0} \frac{\langle V \rangle}{I} = R_N \left\{ I_0 \left[ \frac{\gamma_0(T)}{2} \right] \right\}^{-2}
\]

\[
\gamma_0(T) = \frac{2E_{J0}(T)}{k_B T} = \frac{\Phi_0 l_c(T)}{\pi k_B T}
\]

\( E_{J0}/k_B T \gg 1 \): approximate Bessel function \( \Rightarrow \)

\[
I_0(x) = e^x / 2\pi \sqrt{x}
\]

\[
\frac{R_p(T)}{R_N} \propto E_{J0} \exp \left( -\frac{2E_{J0}}{k_B T} \right)
\]

or

\[
\langle \dot{\phi} \rangle \propto \frac{2e l_c R_N}{\hbar} \exp \left( -\frac{2E_{J0}}{k_B T} \right) = \omega_c \exp \left( -\frac{2E_{J0}}{k_B T} \right)
\]

*attempt frequency is characteristic frequency* \( \omega_c \)

plasma frequency has to be replaced by frequency of overdamped oscillator:

\[
\omega_A = \omega_p \sqrt{\beta_c} = \omega_p \sqrt{\omega_c R_N C} = \omega_c
\]

*washboard potential*: phase diffuses over barrier \( \Rightarrow \) activated nonlinear resistance
3.4.2 Overdamped Junctions: The Ambegaokar-Halperin Theory

Example: $YBa_2Cu_3O_7$ grain boundary Josephson junctions

→ strong effect of thermal fluctuations due to high operation temperature

$YBa_2Cu_3O_7$ grain boundary Josephson junctions

$\rightarrow$ strong effect of thermal fluctuations due to high operation temperature

3.4.2 Overdamped Junctions: The Ambegaokar-Halperin Theory

high temperature superconducting grain boundary Josephson junctions

- $\text{La}_{2-x}\text{Ce}_x\text{CuO}_4$ on $\text{SrTiO}_3$
- optical lithography

3.4.2 Overdamped Junctions: The Ambegaokar-Halperin Theory

overdamped $\text{YBa}_2\text{Cu}_3\text{O}_7$ grain boundary Josephson junction

thermally activated phase slippage

$\Rightarrow$ determination of $I_c(T)$ close to $T_c$

3.4.2 Overdamped Junctions: The Ambegaokar-Halperin Theory

**thermally activated phase slippage**

→ rounding of IVC at \( I \approx I_c \)

rounding decreases with increasing \( \gamma_0 (\equiv \text{increasing } U_0) \)

\[
\gamma_0(T) = \frac{2E_{J0}(T)}{k_BT} = \frac{\Phi_0 I_c(T)}{\pi k_BT}
\]

**analytical** IVC for strong damping \((\beta_c \ll 1)\): (Ambegaokar, Halperin)

\[
\langle V \rangle = \frac{2I_c R_N}{\gamma_0} e^{\pi \gamma_0 i} - 1 \left\{ \int_0^{2\pi} d\varphi e^{-\gamma_0 i\varphi/2} I_0 \left( \gamma_0 \sin \frac{\varphi}{2} \right) \right\}^{-1}
\]

for **small** junctions \((L < \lambda_j)\):

close to \( T_c \): measurement of \( R_p \) at const \( T \) \( \rightarrow I_c(B) \):

\[
R_p(B) = R_N \left\{ I_0 \left[ \frac{\gamma_0(B)}{2} \right] \right\}^{-2}
\]

\[
\gamma_0(B) = \frac{2E_{J0}(B)}{k_BT} = \frac{\Phi_0 I_c(B)}{\pi k_BT}
\]

3.5 Voltage State of Extended Josephson Junctions

3.5.1 Negligible Screening Effects

neglect self-fields $\Rightarrow B = B^{\text{ex}}$ (valid for short junctions)
junction voltage $V = \text{applied voltage } V_0$
$\Rightarrow$ gauge invariant phase difference:

\[
\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} V_0 = \omega_0
\]
\[
\frac{\partial \varphi(z, t)}{\partial z} = \frac{2\pi}{\Phi_0} t_B B_y(z, t)
\]
\[
\Rightarrow \varphi(z, t) = \varphi_0 + \omega_0 t + \frac{2\pi}{\Phi_0} B_y t_B z = \varphi_0 + \omega_0 t + k \cdot z
\]
\[
\Rightarrow J_s(z, t) = J_c \sin (\omega_0 t + k \cdot z + \varphi_0)
\]

$\Rightarrow$ Josephson vortices are moving in $z$-direction with velocity

\[
v_z = \frac{\omega_0}{k} = \frac{V_0}{B_y t_B}
\]
3.5.2 The Time Dependent Sine-Gordon Equation

Long junctions ($L >> \lambda_J$):

**effect of Josephson currents has to be taken into account**

→ magnetic flux density = sum of external and self-generated field

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \]

with $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \varepsilon_0 \mathbf{E}$:

in contrast to static case, now $\frac{\partial \mathbf{E}}{\partial t} \neq 0$

consider 1D junction extending in z-direction, $B = B_y$, current flow in x-direction

\[ \frac{\partial B_y(z, t)}{\partial z} = -\mu_0 J_x(z, t) - \varepsilon_0 \mu_0 \frac{\partial E_x(z, t)}{\partial t} \]

\[ \Rightarrow \frac{\partial^2 \varphi(z, t)}{\partial z^2} = -\frac{2\pi}{\Phi_0} t_B \left\{ \mu_0 J_x(z, t) + \varepsilon_0 \mu_0 \frac{\partial E_x(z, t)}{\partial t} \right\} \]

with $E_x = -V/d$, $J_x = -J_c \sin \varphi$ and $\frac{\partial \varphi}{\partial t} = \frac{2\pi V}{\Phi_0}$:

\[ \frac{\partial^2 \varphi(z, t)}{\partial z^2} = \frac{2\pi t_B \mu_0 J_c}{\Phi_0} \sin \varphi(z, t) + \frac{\varepsilon_0 \mu_0 t_B}{d} \frac{\partial^2 \varphi(z, t)}{\partial t^2} \]

\[ \lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \]

(Josephson penetration depth)

\[ \overline{c} = \sqrt{\frac{d}{\varepsilon_0 \mu_0 t_B}} \]

(propagation velocity)
3.5.2 The Time Dependent Sine-Gordon Equation

\[ 0 = \frac{\partial^2 \varphi(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi(z, t)}{\partial t^2} - \frac{1}{\lambda_J^2} \sin \varphi(z, t) \]

with the Swihart velocity:

\[ \bar{c} = \sqrt{\frac{d}{\varepsilon \varepsilon_0 \mu_0 t_B}} = \frac{1}{\sqrt{\varepsilon \varepsilon_0 \mu_0}} \sqrt{\frac{d}{\varepsilon (2\lambda_L + d)}} = c \sqrt{\frac{1}{\varepsilon (1 + 2\lambda_L/d)}} \]

\[ \lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 t_B J_c}} \]

\[ \equiv \text{velocity of TEM mode in the junction transmission line} \]

example: \( \varepsilon \approx 5-10, 2\lambda_L/d \approx 50-100 \rightarrow \text{Swihart velocity} \approx 0.1 \cdot c \)

\[ \rightarrow \text{reduced wavelength:} \]

\( \text{e.g. } f = 10 \text{ GHz: free space: 3 cm, in junction: 1 mm} \)

other form of time-dependent Sine-Gordon equation

\[ \lambda_J^2 \frac{\partial^2 \varphi(z, t)}{\partial z^2} - \frac{4\pi^2}{\omega_p^2} \frac{\partial^2 \varphi(z, t)}{\partial t^2} - \sin \varphi(z, t) = 0 \]

\[ \omega_p^2 = 2e I_c / \hbar C \quad C/A_i = \varepsilon \varepsilon_0 / d \quad I_c/A_i = J_c \quad c^2 = 1/\varepsilon_0 \mu_0 \quad \Rightarrow \omega_p/2\pi = \bar{c}/\lambda_J \]
3.5.2 The Time Dependent Sine-Gordon Equation

time-dependent Sine-Gordon equation:

\[ \lambda_j \frac{\partial^2 \varphi(z, t)}{\partial z^2} - \frac{4\pi^2}{\omega_p^2} \frac{\partial^2 \varphi(z, t)}{\partial t^2} - \sin \varphi(z, t) = 0 \]

mechanical analogue:

*chain of mechanical pendula*

attached to a twistable rubber ribbon

Note: short junction w/o magnetic field: \( \frac{\partial^2 \varphi}{\partial z^2} = 0 \)

\[ \rightarrow \text{rigid connection of pendula} \]

\[ \rightarrow \text{corresponds to single pendulum} \]
3.5.3 Solutions of the Time Dependent SG Equation

- **simple cases:**
  1-dimensional junction \( W \ll \lambda_J \), short and long junctions

(i) **short junctions** \( L \ll \lambda_J \) @ low damping
  \rightarrow \text{neglect } z\text{-variation of } \varphi:

\[
\frac{\partial^2 \varphi(z, t)}{\partial t^2} + \frac{\omega_p^2}{4\pi^2} \sin \varphi(z, t) = 0
\]

\rightarrow \text{equivalent to } RCSJ \text{ model for } G = 0, I = 0
small amplitudes \rightarrow \text{plasma oscillations}
(oscillation of } \varphi \text{ around minimum of washboard potential)

(ii) **long junctions** \( L \gg \lambda_J \), solitons
  \rightarrow \text{solution for infinitely long junction } \rightarrow \text{soliton or fluxon:}

\[
\varphi(z, t) = 4 \arctan \left\{ \exp \left( \pm \frac{z - z_0}{\lambda_J} - \frac{v_z t}{c} \right) \right\}
\]

\( \varphi = \pi \) at \( z = z_0 + v_z t \)
goes from 0 to \( 2\pi \) for \( z \) going from \( -\infty \) to \( \infty \)
\rightarrow \text{fluxon (antifluxon: } \infty \text{ to } -\infty)
3.5.3 Solutions of the Time Dependent SG Equation

\[ \varphi = \pi \text{ at } z = z_0 + v_z t \]
goes from 0 to \(2\pi\) for \(z\) from \(-\infty\) to \(\infty\)
\(\rightarrow\) fluxon (antifluxon: \(\infty\) to \(-\infty\))

**pendulum analog:**
local \(360^\circ\) twist of rubber ribbon

applied current \(\rightarrow\) **Lorentz force** \(\rightarrow\) *motion of phase twist (fluxon)*

- fluxon as particle: **Lorentz contraction** for \(v_z \rightarrow \bar{c}\)
- local change of phase difference \(\rightarrow\) voltage
  
  **moving** fluxon \(\equiv\) voltage **pulse**

- other solutions: fluxon-fluxon collisions, ...
3.5.3 Solutions of the Time Dependent SG Equation

- Linearized Sine-Gordon equation: Josephson plasma waves

Let:
\[
\varphi(z, t) = \varphi_0(z) + \varphi_1(z, t)
\]

\(\varphi_1\): small deviation

\[\Rightarrow\] approximation:
\[
\sin \varphi \simeq \sin \varphi_0 + \varphi_1 \cos \varphi_0
\]

Substitution (keeping only linear terms):
\[
\frac{\partial^2 \varphi_0}{\partial z^2} + \frac{\partial^2 \varphi_1(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi_1(z, t)}{\partial t^2} - \frac{1}{\lambda_J^2} \sin \varphi_0 - \frac{1}{\lambda_J^2} \cos \varphi_0 \varphi_1(z, t) = 0
\]

\(\varphi_0\) solves time independent SG, \(\frac{\partial^2 \varphi_0}{\partial z^2} = \sin \varphi_0 / \lambda_J:\)

\[
\frac{\partial^2 \varphi_1(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi_1(z, t)}{\partial t^2} - \frac{1}{\lambda_J^2} \cos \varphi_0 \varphi_1(z, t) = 0
\]

Solution:
\[
\varphi_1(z, t) = \exp(-i[kz - \omega t])
\]

(Small amplitude plasma waves)

Dispersion relation \(\omega(k):\)
\[
\omega^2 = \frac{c^2 k^2}{\lambda_J^2} + \omega_{p, J}^2
\]

Josephson plasma frequency:
\[
\frac{\omega_{p, J}^2}{4\pi^2} = \frac{c^2}{\lambda_J^2} \cos \varphi_0
\]
3.5.3 Solutions of the Time Dependent SG Equation

for $\omega > \omega_{p,j}$: mode propagation

*pendulum analogue*: deflect one pendulum $\rightarrow$ relax $\rightarrow$ wave like excitation

for $\omega = \omega_{p,j}$: infinite wavelength *Josephson plasma wave* (typically $\sim 10$ GHz)

- plane waves:

$$\Rightarrow \frac{\partial^2 \varphi(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi(z, t)}{\partial t^2} - \frac{1}{\lambda_j^2} \sin \varphi(z, t) = 0$$

for very large $\lambda_j$ or very small $I$:

$\rightarrow$ neglect $\sin \varphi/\lambda_j^2$ term

$\rightarrow$ linear wave equation
3.5.4 Resonance Phenomena

interaction of fluxons/plasma waves with oscillating Josephson current

\[ \rightarrow \text{rich variety of interesting resonance phenomena} \]

(i) flux-flow steps and the Eck peak

for \( B_{\text{ext}} > 0 \):
- spatially modulated Josephson current density \( \rightarrow \) moves at \( v_z = V / B_y t_b \)
- Josephson current can excite Josephson plasma waves
  resonance: em waves couple strongly to Josephson current if \( \bar{c} = v_z \)

corresponding junction voltage:

\[ V_{\text{Eck}} = \bar{c} B_y t_B = \sqrt{\frac{d}{\epsilon \epsilon_0 \mu_0 t_B}} B_y t_B = \frac{\omega_p}{2\pi} \frac{\lambda_j}{L} B_y t_B L = \frac{\omega_p}{2\pi} \frac{\lambda_j}{L} \Phi_0 \frac{\Phi}{\Phi_0} \]

\[ \rightarrow \text{Eck peak at frequency:} \quad \omega_{\text{Eck}} = \frac{2e}{\hbar} V_{\text{Eck}} = \omega_p \frac{\lambda_j}{L} \frac{\Phi}{\Phi_0} \]

\[ \bar{c} = \frac{\omega_p}{2\pi} \frac{\lambda_j}{L} \]
\[ \Phi = B_y t_B L \]

traveling current wave only excites traveling em wave of same direction
- @ low damping, short junctions: em wave is reflected at open end
  \( \rightarrow \) Eck peak only observed in long junctions at medium damping
### 3.5.4 Resonance Phenomena

**other point of view:**

- Lorentz force $\rightarrow$ Josephson vortices move at $v_z = \frac{V_{ff}}{B_y t_B}$
- increase driving force $\rightarrow$ increase $v_z$
- maximum possible speed: $v_z = \bar{c}$

$\Rightarrow$ step in IVC: *flux flow step*

\[
V_{ffs} = \bar{c} B_y t_B = \bar{c} \frac{\Phi}{L} = \frac{\omega_p}{2\pi} \frac{\lambda_J}{L} \frac{\Phi_0}{\Phi_0}
\]

$\Rightarrow$ corresponds to Eck voltage

\[
\bar{c} = \frac{\omega_p}{2\pi} \lambda_J
\]
3.5.4 Resonance Phenomena

(ii) Fiske steps

Standing *em waves* in junction “cavity” @ $\omega_n = 2\pi f_n = 2\pi \frac{c}{2L} n = \frac{\pi c}{L} n$

$V_n = \frac{\hbar}{2e} \omega_n = \Phi_0 \frac{c}{2L} n = \frac{\omega_p}{2\pi} \frac{\lambda_J}{L} \Phi_0 \frac{n}{2}$

**Standing wave pattern of em wave and Josephson current match $\Rightarrow$ steps in IVC**

for $L \sim 100 \mu m$

first Fiske step $\sim 10$ GHz

wave length of Josephson current density $2\pi/k \propto B$

resonance condition $L = \frac{c}{2f_n} n = \frac{\lambda}{2} n \Rightarrow kL = n\pi$ or $\Phi = n\Phi_0/2$

here: **maximum** Josephson current of short junction *vanishes*

damping of standing wave pattern by dissipative effects

$\Rightarrow$ broadening of Fiske steps

$\Rightarrow$ observation only for *small and medium damping*
3.5.4 Resonance Phenomena

**Fiske steps** at small damping and/or small magnetic field

**Eck peak** at medium damping and/or medium magnetic field

For voltages \( \neq V_{\text{Eck}} \) or \( V_n \):

\[
J_s(z, t) = J_C \sin (\omega_0 t + k \cdot z + \varphi_0)
\]

\[
\langle I_s \rangle \approx 0 \rightarrow I = I_N(V) = V / R_N(V)
\]
3.5.4 Resonance Phenomena

(iii) zero field steps

- motion of trapped flux due to Lorentz force (w/o magnetic field)
- junction of length $L$, moving back and forth: $T = 2L/v_z$, phase change: $4\pi$
- at large bias currents:

$$V_{zfs} = \frac{\Phi}{2e} = \frac{4\pi \hbar}{T 2e} = \frac{4\pi \hbar}{2L/c 2e} = \frac{h c}{2e L} = \frac{\omega_p \lambda_J}{\pi L} \Phi_0$$

$n$ fluxons: $V_{n,zfs} = n \cdot V_{zfs}$

$V_{n,zfs} = 2 \times$ Fiske voltage $V_n$ (fluxon has to move back and forth)

$V_{fzs} = V_{n,zfs}$ for $\Phi = n \Phi_0$

IVCs of annular Nb/insulator/Pb Josephson junction containing a different number of trapped fluxons

Vortex-Cherenkov radiation $\rightarrow$ lecture notes
Summary (voltage state of short junctions)

voltage state: (Josephson + normal + displacement + fluctuation) current = total current

\[ I = I_c \sin \varphi + G_N(V) V + C \frac{dV}{dt} + I_F \]

\[ \frac{d \varphi}{dt} = \frac{2e V}{\hbar} \]

\[ I = I_c \sin \varphi + G_N(V) \frac{\Phi_0}{2\pi} \frac{d \varphi}{dt} + C \frac{\Phi_0}{2\pi} \frac{d^2 \varphi}{dt^2} + I_F \]

equation of motion for phase difference \( \varphi \):

**RCSJ-model** \((G_N(V) = \text{const.})\):

\[ \beta_C \frac{d^2 \varphi}{d \tau^2} + \frac{d \varphi}{d \tau} + \sin \varphi - i - i_F(\tau) = 0 \]

motion of phase particle in the tilted washboard potential:

\[ U = E_{j0}[1 - \cos \varphi - (I/I_c)\varphi] \]

equivalent circuit: LCR resonator, characteristic frequencies:

\[ \omega_p = \sqrt{\frac{1}{L_c C}} = \sqrt{\frac{2el_c}{\hbar C}} \quad \omega_c = \frac{R}{L_c} = \frac{2el_c R}{\hbar} \quad \omega_{RC} = \frac{1}{RC} \]

quality factor:

\[ Q^2 = \beta_C \equiv \frac{2e}{\hbar} I_c R^2 C \]
Summary (voltage state of short junctions)

IVC: \[ \langle V(t) \rangle = I_c R \sqrt{ \left( \frac{I}{I_c} \right)^2 - 1 } \quad \text{for} \quad \frac{I}{I_c} > 1 \] (strong damping, \( \beta_c \ll 1 \))

driving with \( V(t) = V_{dc} + V_1 \cos \omega_1 t \) \( \Rightarrow \) Shapiro steps at

\[ V_n = n \frac{\Phi_0}{2\pi} \omega_1 \]

amplitudes:

\[ |\langle I_s \rangle_n| = I_c \left| J_n \left( \frac{2\pi V_1}{\Phi_0 \omega_1} \right) \right| \]
Summary (voltage state of long junctions)

Equation of motion: **Sine-Gordon equation**:

\[
\frac{\partial^2 \varphi(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi(z, t)}{\partial t^2} - \frac{1}{\lambda_j^2} \sin \varphi(z, t) = 0
\]

*Swihart velocity* (propagation velocity of em waves)

Prominent solutions: plasma oscillations and solitons

**Nonlinear interactions** of these excitations with Josephson current:
→ *flux-flow steps, Fiske steps, zero-field steps*
3.6 Full Quantum Treatment of Josephson Junctions
Secondary Quantum Macroscopic Effects

- so far: Josephson junction was treated **classically**
  - $\varphi$ and $d\varphi/dt$ \([\propto Q = CV = C \left(\frac{\hbar}{2e}\right) \left(\frac{d\varphi}{dt}\right)\] as **purely classical variables**
  - motion of $\varphi$ is treated analogous to **classical motion of particle in tilted washboard potential**
  - classical energies:
    - potential energy $U(\varphi)$, energy associated with Josephson coupling energy (equivalent circuit: *flux stored in Josephson inductance*)
    - kinetic energy $K(\varphi)$, associated with $\left(\frac{d\varphi}{dt}\right)^2 \propto \frac{Q^2}{2C} = \frac{1}{2}CV^2$
      (equivalent circuit: *charge stored on junction capacitance*)

- current-phase and voltage-phase relation have **quantum origin** (macroscopic quantum model)
  \[ \rightarrow primary \ quantum \ macroscopic \ effects \]

but so far: variables $I, Q, V, \varphi$ are assumed to be **measurable simultaneously**
  \[ \rightarrow more \ precise \ quantum \ theory \ is \ needed \]
  \[ \rightarrow secondary \ quantum \ macroscopic \ effects \]
3.6.1 Quantum Consequences of the small Junction Capacitance

- **validity of classical treatment:**
  - consider an isolated, low-damping junction, $I = 0$
    - cosine potential, depth $2E_{J0}$
    - approx. close to minimum: **harmonic oscillator**, frequency $\omega_p$, level spacing $\hbar \omega_p$

\[
2E_{J0} \approx \hbar \omega_p\]

\[
E_{C} = \frac{e^2}{2C}
\]

\[
\hbar \omega_p = \sqrt{8E_{J0}E_C}
\]

classical treatment valid as long as
\[
\frac{E_{J0}}{\hbar \omega_p} \approx \left(\frac{E_{J0}}{E_C}\right)^{1/2} \gg 1
\]

(\text{level spacing} \ll \text{potential depth})

- $E_C \propto 1/C \propto 1/A$
- $E_{J0} \propto I_c \propto A$

enter **quantum regime** by

**decreasing junction area**
3.6.1 Quantum Consequences of the small Junction Capacitance

numbers:

(i) area \( A = 10 \, \mu m^2 \):
- barrier: \( d = 1 \, nm \), \( \epsilon = 10 \)
- \( J_c = 100 \, A/cm^2 \)
  \( \rightarrow E_{j0} = 3 \times 10^{-21} \, J \)
- \( C = \epsilon \epsilon_0 A/d \simeq 9 \times 10^{-13} \, F \)
  \( \rightarrow E_c \simeq 1.6 \times 10^{-26} \, J \)
  \( \rightarrow \text{classical junction} \)

(ii) area \( A \simeq 0.02 \, \mu m^2 \):
- \( C \simeq 1 \, fF \)
  \( \rightarrow E_c \simeq E_{j0} \)

we also need \( T \ll 100 \, mK \) for \( k_B T \ll E_c \)
3.6.1 Quantum Consequences of the small Junction Capacitance

- consider a strongly **underdamped** junction (negligible damping) with \(d\varphi/dt \neq 0\)

  kinetic energy:

  \[
  K = \frac{1}{2} CV^2 = \frac{Q^2}{2C} = \frac{1}{2} C \left( \frac{\hbar}{2e} \right)^2 \varphi^2 = \frac{1}{2} E_{J0} \frac{\varphi^2}{\omega_p^2}
  \]

  \(\rightarrow\) energy due to **extra charge** \(Q\) on one junction electrode **due to** \(V\)

  total energy:

  \[
  E = K + U = E_{J0} \left( 1 - \cos \varphi + \frac{1}{2} \frac{\varphi^2}{\omega_p^2} \right)
  \]

  \(U(\varphi) \propto 1 - \cos \varphi:\) potential energy

  \(K(\varphi) \propto \varphi^2:\) kinetic energy

- consider \(E\) as **junction Hamiltonian**, rewrite kinetic energy:

  \[
  K = \frac{Q^2}{2C} = \frac{1}{2} \frac{1}{(\hbar/2e)^2 C} \left( \frac{\hbar}{2e} \right)^2 Q^2
  \]

  \(\rightarrow\) \(p = \left( \frac{\hbar}{2e} \right) Q\)
3.6.1 Quantum Consequences of the small Junction Capacitance

**canonical quantization**, operator replacement:

\[ \frac{\hbar}{2e} Q \rightarrow -i\hbar \frac{\partial}{\partial \varphi} \]

with \( N = Q/2e \): # of Cooper pairs:

\[ Q = -i2e \frac{\partial}{\partial \varphi} \quad N = -i \frac{\partial}{\partial \varphi} \]

we get the **Hamiltonian**:

\[ \mathcal{H} = \frac{Q^2}{2C} + E_{J0}(1 - \cos \varphi) = -\frac{(2e)^2}{2C} \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi) \]

\[ \mathcal{H} = -4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi) \]

→ describes only Cooper pairs

\( E_C = e^2 / 2C \): charging energy for a **single** electron charge

**commutation rules** for the operators:

\[ [\varphi, Q] = i2e \quad ; \quad [\varphi, N] = i \quad \text{or} \quad [\varphi, \frac{\hbar}{2e} Q] = i\hbar \]

\( N \equiv Q/2e \): deviation of # of CP in electrodes from equilibrium

**uncertainty relation**:

\[ \Delta N \cdot \Delta \varphi \geq 1 \]
3.6.1 Quantum Consequences of the small Junction Capacitance

- Hamiltonian in **flux basis** $\phi = \frac{\hbar}{2e} \varphi = \frac{\Phi_0}{2\pi} \varphi$:

$$
\mathcal{H} = \frac{Q^2}{2C} + E_{j0} \left( 1 - \cos 2\pi \frac{\phi}{\Phi_0} \right) = -\frac{(2e)^2}{2C} \frac{\hbar^2}{(2e)^2} \frac{\partial^2}{\partial \phi^2} + E_{j0} \left( 1 - \cos 2\pi \frac{\phi}{\Phi_0} \right)
$$

**commutator**: $[\phi, Q] = i\hbar$

$\Rightarrow$ $Q$ and $\phi$ are *canonically conjugate* (cf. $x$ and $p$),

deviations from “classical” description:

$\Rightarrow$ *secondary quantum macroscopic effects*
3.6.2 Limiting Cases: The Phase and Charge Regime

- **the phase regime**: $\hbar \omega_p \ll E_J, E_c \ll E_J$ (phase $\varphi$ is good quantum number)

  lowest energy levels: **localized near bottom** of potential wells at $\varphi_n = 2\pi n$

  Taylor series for $U(\varphi) \rightarrow$ **harmonic oscillator**,
  frequency $\omega_p$, eigenenergies: $E_n = \hbar \omega_p \left( n + \frac{1}{2} \right)$

  $\rightarrow$ **ground state**: **narrowly peaked** wave function at $\varphi = \varphi_n$
  $\rightarrow$ **small** phase fluctuations $\Delta \varphi$
  $\rightarrow$ **large** fluctuations of $Q$ on electrodes since $\Delta Q \cdot \Delta \varphi \geq 2e$
  **small** $E_c \rightarrow$ pairs can easily fluctuate, large $\Delta Q$

  $\rightarrow$ negligible $\Delta \varphi \Rightarrow$ **classical** treatment of **phase dynamics** is good approximation
3.6.2 Limiting Cases: The Phase and Charge Regime

- **the phase regime:** \( \hbar \omega_p \ll E_J, E_c \ll E_J \) (phase \( \varphi \) is good quantum number)

Hamiltonian:
\[
\mathcal{H} = -4E_c \frac{\partial^2}{\partial \varphi^2} + E_J(1 - \cos \varphi)
\]

define: \( a = (E - E_J)/E_c, \ b = E_J/2E_c \) and \( z = \varphi/2 \)

\[\rightarrow\text{ Mathieu equation:}\]
\[\frac{\partial^2 \psi}{\partial z^2} + (a + 2b \cos 2z) \psi = 0\]

general solution:
\[\psi(\varphi) = \sum_q c_q \psi_q\]

**Bloch waves:**
\[\psi_q(\varphi) = u_q(\varphi) \exp(iq\varphi) \quad \text{with} \quad u_q(\varphi) = u_q(\varphi + 2\pi)\]

\( q \): charge/pair number variable, \( q \) is continuous (cf. charge on capacitor)
\[\rightarrow\] \( \Psi \) is not \( 2\pi \)-periodic

1-dimensional problem \( \rightarrow \) numerical solution

known from **periodic potential** problem in solid state physics
\[\rightarrow \text{ energy bands}\]
3.6.2 Limiting Cases: The Phase and Charge Regime

Variational approach for approximate ground state

trial function for $E_C \ll E_J$: $\Psi(\varphi) \propto \exp \left( -\frac{\varphi^2}{4\sigma^2} \right)$

(choose $\sigma$ to find) minimum energy:

$$E_{\min} = E_J \left( 1 - \left[ 1 - \sqrt{\frac{2E_C}{E_J}} \right]^2 \right) = E_J \left( 1 - \left[ 1 - \frac{\hbar \omega_p}{2E_J} \right]^2 \right)$$

$$\frac{E_C}{E_J} = 0.1$$

$E_{\min} = 0.1 E_J$

Tunneling coupling $\propto \exp \left( -\frac{2E_J - E}{\hbar \omega_p} \right) \rightarrow$ very small since $\hbar \omega_p \ll E_J$

$\rightarrow$ tunneling splitting of low lying states is exponentially small
3.6.2 Limiting Cases: The Phase and Charge Regime

- **the charge regime**: \( \hbar \omega_p \gg E_{J0}, E_c \gg E_{J0} \) (charge \( Q \) is good quantum number)

kinetic energy \( \propto E_c \left( \frac{d\varphi}{dt} \right)^2 \) is dominating, complete delocalization of phase

\( \rightarrow \) wave function should approach constant value, \( \Psi(\varphi) \approx \text{const} \).

\( \rightarrow \) large phase fluctuations, small charge fluctuations: \( \Delta Q \cdot \Delta \varphi \geq 2e \)

\( \rightarrow \) charge \( Q \) (corresponds to momentum) is good quantum number

Hamiltonian:

\[
\mathcal{H} = -4E_c \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi)
\]

appropriate trial function:

\( \psi(\varphi) \propto (1 - \alpha \cos \varphi) \quad \alpha \ll 1 \)

approximate ground state energy

\[
E_{\text{min}} \approx E_{J0} \left(1 - \frac{E_{J0}}{8E_c}\right) = E_{J0} \left(1 - \frac{E_{J0}^2}{(\hbar \omega_p)^2}\right)
\]

second order in \( E_{J0} \)
3.6.2 Limiting Cases: The Phase and Charge Regime

- **the charge regime**: \( \hbar \omega_p \gg E_J_0, \ E_c \gg E_J_0 \) (charge \( Q \) is good quantum number)

\[
E_{\text{min}} \simeq E_J_0 \left(1 - \frac{E_J_0}{8E_c}\right) = E_J_0 \left(1 - \frac{E_J_0^2}{(\hbar \omega_p)^2}\right)
\]

\[
\frac{E_c}{E_J_0} = 2.5
\Rightarrow E_{\text{min}} = 0.95 \ E_J_0
\]

- periodic potential is **weak** \( \Rightarrow \) **strong coupling** between neighboring phase states
- broad bands
- compare to electrons moving in strong (**phase regime**) or weak (**charge regime**) periodic potential of a crystal
Brief Summary

classical description valid for:
level spacing $\ll$ potential depth

$$\frac{\hbar \omega_p}{E_J_0} = \sqrt{\frac{8E_C}{E_J_0}} \ll 1$$

$E_C \propto 1/C \propto 1/A$
$E_{J_0} \propto I_c \propto A$

the phase regime: $\hbar \omega_p \ll E_{J_0}, \ E_C \ll E_{J_0}$

$\rightarrow$ small phase fluctuations
$\rightarrow$ large charge fluctuations

the charge regime: $\hbar \omega_p \gg E_{J_0}, \ E_C \gg E_{J_0}$

$\rightarrow$ large phase fluctuations
$\rightarrow$ small charge fluctuations
3.6.3 Coulomb and Flux Blockade

- **Coulomb blockade in normal metal tunnel junctions:**
  - voltage: $V \rightarrow$ charge: $Q = CV$,
  - energy: $E = Q^2 / 2C$

- **single electron tunneling:** charge on one electrode changes to $Q - e$
  - electrostatic energy: $E' = (Q - e)^2 / 2C$
  - tunneling only *allowed* for $E' \leq E$
  - we need $|Q| \geq e/2$ or: $|V| \geq V_{CB} = V_c = e/2C \rightarrow$ *Coulomb blockade*

- **thermal fluctuations:** $E_C = \frac{e^2}{2C} > k_B T \Rightarrow C < \frac{e^2}{2k_B T}$ *(small thermal fluctuations)*
  - numbers: $C \approx 1$ fF @ 1 K, for $d = 1$ nm and $\varepsilon = 5 \rightarrow A \approx 0.02$ µm²

- **quantum fluctuations:** $\Delta E \cdot \Delta t \geq \hbar$:
  - finite tunnel resistance $\Rightarrow \tau_{RC} = RC$ *(decay of charge fluctuations)*
  - $\Delta t = 2\pi RC$, $\Delta E = e^2 / 2C$
  - $R \geq \frac{h}{e^2} = R_K = 24.6$ kΩ *(small quantum fluctuations)*

- **Coulomb blockade in superconducting tunnel junction:**
  - for $\frac{Q^2}{2C} > k_B T, eV$ ($Q = 2e$) $\Rightarrow$ no flow of Cooper pairs
  - **threshold voltage:** $|V| \geq V_{CB} = V_c = \frac{2e}{2C} = \frac{e}{C}$

- Coulomb blockade $\Rightarrow$ charge is fixed, *phase is completely smeared out*
3.6.3 Coulomb and Flux Blockade

- phase or flux blockade in a Josephson junction:

  \[ \text{current: } I \rightarrow \text{flux: } \Phi = L I, \]

  \[ \text{energy: } E = \Phi^2 / 2L \]

  phase is blocked due to large \( E_J = \Phi_0 I_c / 2\pi \)

  - \( I_c \) takes the role of \( V_{CB} \)
  - phase change of \( 2\pi \) equivalent to flux change change of \( \Phi_0 \)

  \[ \rightarrow \text{flux blockade} \quad |I| \geq I_{FB} = I_c = \frac{(\Phi_0 / 2\pi)}{L_c} \]

  cf. charge blockade \[ |V| \geq V_{CB} = V_c = \frac{e}{C} \]

  analogy: \( I \leftrightarrow V, \quad 2e \leftrightarrow \frac{\Phi_0}{2\pi}, \quad C \leftrightarrow L \)

- fluctuations, we need:

  \[ E_J \gg k_B T \]

  and

  \[ \Delta E \cdot \Delta t \geq \hbar: \quad \text{with } \Delta t = 2\pi L / R \text{ and } \Delta E = 2E_J \rightarrow \quad R \leq \frac{\hbar}{(2e)^2} = \frac{1}{4} R_K \]
3.6.4 Coherent Charge and Phase States

coherent charge states
island charge continuously changed by gate

for independent charge states:

parabola: \[ E = (Q - n \cdot 2e)^2 / 2C \Sigma \]

for \( E_{J0} > 0 \): interaction of \( |n\rangle \) and \( |n + 1\rangle \) at the crossing points \( Q = (n + \frac{1}{2}) \cdot 2e \)

→ coherent superposition states: \[ \Psi_{\pm} = a|n\rangle \pm b|n + 1\rangle \]

→ splitting of charge energy at crossing points \( \equiv \) level anti-crossing

→ splitting magnitude \( \propto \) Josephson coupling energy \( E_{J0} \)
3.6.4 Coherent Charge and Phase States

average charge on the island as a function of the applied gate voltage

coherent superposition of charge states: experiment by Nakamura, Pashkin, Tsai
3.6.4 Coherent Charge and Phase States

→ coherent superposition states: \( \Psi_{\pm} = a|n\rangle \pm b|n+1\rangle \)

→ splitting of charge energy at crossing points \( \equiv \text{level anti-crossing} \)

→ splitting magnitude \( \Rightarrow \) Josephson coupling energy
3.6.4 Coherent Charge and Phase States

coherent phase states
→ interaction of two adjacent phase states
→ e.g. rf-SQUID

magnetic energy of flux $\phi = \left( \frac{\Phi_0}{2\pi} \right) \varphi$
in the ring

$$U(\phi) = \frac{(\phi - \phi_{\text{ext}})^2}{2L} + E_{j0} \left( 1 - \cos 2\pi \frac{\phi}{\Phi_0} \right)$$

$\Phi_{\text{ext}} = \Phi_0/2$

tunnel coupling: $\psi_{\pm} = a|L\rangle \pm b|R\rangle$

*experimental evidence* for quantum coherent superposition of states
3.6.5 Quantum Fluctuations

Violation of conservation of energy on small time scales, obey $\Delta E \cdot \Delta t \geq \hbar$

\[ S_I(f) = 2\pi S_I(\omega) = \frac{4 E(\omega, T)}{R_N} \]

\[ E(\omega, T) = \frac{\hbar \omega}{2} + \hbar \omega \frac{1}{\exp \left( \frac{\hbar \omega}{k_B T} \right) - 1} = \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \]

Vacuum fluctuations of oscillator (Planck distribution)

Transition from “thermal” Johnson-Nyquist ($\hbar \omega, eV \ll k_B T$) noise to quantum noise:

**Classical limit** ($\hbar \omega, eV \ll k_B T$): $S_I(\omega) = \frac{1}{2\pi} \frac{4k_B T}{R_N}$

**Quantum limit** ($\hbar \omega, eV \gg k_B T$): $S_I(\omega) = \frac{1}{2\pi} \frac{2\hbar \omega}{R_N} = \frac{1}{2\pi} \frac{2eV}{R_N}$
**3.6.6 Macroscopic Quantum Tunneling**

*escape* of the “phase particle” from minimum of washboard potential by **tunneling**

→ **macroscopic**: phase difference is tunneling (**collective state**)

→ states **easily** **distinguishable**

competing process: **thermal activation**

→ **low temperatures**

neglect damping

dc-bias: term $-\hbar I\phi/2e$ in Hamiltonian

**curvature** at potential minimum:

$$\frac{\partial^2 U}{\partial \phi^2} = E_{J0} \sqrt{1 - i^2} \quad i = I/I_c$$

(classical) **small oscillation frequency**:

$$\omega_A = \omega_p (1 - i^2)^{1/4} \quad \text{(attempt frequency)}$$
3.6.6 Macroscopic Quantum Tunneling

quantum mechanically:

- **tunnel** coupling of bound states to outgoing waves → continuum of states
  - but: only states corresponding to quasi-bound states have high amplitude
  - → in well states of width $\Gamma = \hbar/\tau$ ($\tau$: lifetime for escape)

determination of wave functions

- **wave matching method**
  - **exponential prefactor** within WKB approximation
  - → decay in barrier:

$$
|\psi(x)|^2 \propto \exp \left\{ -\frac{2}{\hbar} \int_{ll} \sqrt{2M[V(x) - E]} \, dx \right\}
$$

decay of wave function of particle with mass $M$ and energy $E$

for $U(\varphi) \gg E_0 = \hbar \omega_A/2$

$$
|\psi(\varphi)|^2 \propto \exp \left\{ -\frac{2}{\hbar} \int_{ll} \sqrt{2 \left( \frac{\hbar}{2e} \right)^2 C \left[ U(\varphi) - \frac{\hbar \omega_A}{2} \right]} \, d\varphi \right\}
$$

mass effective barrier height
3.6.6 Macroscopic Quantum Tunneling

constant barrier height:

\[ |\psi(\varphi)|^2 \propto \exp \left\{ -\sqrt{\frac{U_0}{E_C}} \Delta \varphi \right\} \]

escape rate:

\[ \Rightarrow \Gamma = \frac{\omega_A}{2\pi} \exp \left\{ -\sqrt{\frac{U_0}{E_C}} \Delta \varphi \right\} \]

increasing bias current:

\[ U_0 \simeq 2E_{j0}(1 - i^2)^{3/2} \quad \text{and} \quad \Delta \varphi \simeq \pi \sqrt{1 - i^2} \]

\( U_0 \) decreases with \( i \to \Gamma \) becomes measurable

temperature \( T^* \) where \( \Gamma_{tunnel} = \Gamma_{TA} \simeq \exp \left( -\frac{U_0}{k_BT} \right) \)

for \( I \simeq 0 \):

\[ U_0 \simeq 2E_{j0} \quad \hbar \omega_p = \sqrt{8E_{j0}E_C} \simeq 2\sqrt{U_0E_C} \quad \Delta \varphi \simeq \pi \]

\[ \Rightarrow \Gamma = \frac{\omega_p}{2\pi} \exp \left\{ -2\pi \frac{U_0}{\hbar \omega_p} \right\} \]

\[ \Rightarrow k_B T^* \simeq \frac{\hbar \omega_p}{2\pi} \]

for \( I > 0 \):

\[ \sqrt{U_0} \propto (1 - i^2)^{3/4} \quad \Delta \varphi \propto (1 - i^2)^{1/2} \]

\[ \Rightarrow k_B T^* \simeq \frac{\hbar \omega_A}{2\pi} = \frac{\hbar \omega_p}{2\pi} (1 - i^2)^{1/4} \]

for \( \omega_p \approx 10^{11} \text{ s}^{-1} \)

\[ \Rightarrow T^* \sim 100 \text{ mK} \]
3.6.6 Macroscopic Quantum Tunneling

additional topic: effect of damping

coupling of the system to the environment (heat bath): see e.g. Caldeira & Leggett
damping suppresses MQT

crossover temperature:

\[ k_B T^* \approx \frac{\hbar \omega_R}{2\pi} \quad \text{with} \quad \omega_R = \omega_A \left\{ \sqrt{1 + \alpha^2} - \alpha \right\}, \quad \alpha = \frac{1}{2R_N C \omega_A} \]

for \( \alpha \gg 1 \): \( \omega_R \ll \omega_A \rightarrow \text{lower } T^* \)

quantum junction: lightly damped
classical junction: moderately damped

phase diffusion by MQT
see lecture notes
Brief Summary

- Coulomb/flux **blockade**
- **coherent** charge and phase states
- macroscopic quantum tunneling
- effects of dissipation
Summary (secondary quantum macroscopic effects)

classical description only in the phase regime (large junctions): \( E_C \ll E_{J0} \)

for \( E_C \gg E_{J0} \): quantum description (negligible damping):

\[
\mathcal{H} = 4E_C \frac{\partial^2}{\partial \varphi^2} + E_{J0}(1 - \cos \varphi)
\]

phase difference \( \varphi \) and Cooper pair number \( N = Q/2e \) are canonically conjugate variables:

\[
[\varphi, \frac{\hbar}{2e}Q] = i\hbar \implies \Delta N \cdot \Delta \varphi \geq 1
\]

phase regime: \( \Delta \varphi \to 0 \) and \( \Delta N \to \infty \)

charge regime: \( \Delta N \to 0 \) and \( \Delta \varphi \to \infty \)

charge regime at \( T = 0 \): Coulomb blockade:

\( V_{CB} \geq e/C \)

flux regime at \( T = 0 \): flux blockade:

\( I_{FB} \geq \Phi_0/2\pi L_c \)

at \( I < I_c \): escape out of the washboard by:

thermal activation
tunneling (macroscopic quantum tunneling)

crossover between TA and tunneling:

\[
k_B T^* \simeq \frac{\hbar \omega_A}{2\pi} = \frac{\hbar \omega_p}{2\pi} \left[ 1 - \left( \frac{1}{l_c} \right)^2 \right]^{1/4}
\]